

# KMS STATES ON GENERALISED BUNCE–DEDDENS ALGEBRAS AND THEIR TOEPLITZ EXTENSIONS

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**ABSTRACT.** We study the generalised Bunce–Deddens algebras and their Toeplitz extensions constructed by Kribs and Solel from a directed graph and a sequence  $\omega$  of positive integers. We describe both of these  $C^*$ -algebras in terms of novel universal properties, and prove uniqueness theorems for them; if  $\omega$  determines an infinite supernatural number, then no aperiodicity hypothesis is needed in our uniqueness theorem for the generalised Bunce–Deddens algebra. We calculate the KMS states for the gauge action in the Toeplitz algebra when the underlying graph is finite. We deduce that the generalised Bunce–Deddens algebra is simple if and only if it supports exactly one KMS state, and this is equivalent to the terms in the sequence  $\omega$  all being coprime with the period of the underlying graph.

## 1. INTRODUCTION

Every Cuntz–Krieger algebra  $\mathcal{O}_A$  carries a gauge action of  $\mathbb{T}$  which lifts to an action  $\alpha$  of  $\mathbb{R}$ . Enomoto, Fujii and Watatani [6] proved that when  $A$  is irreducible,  $(\mathcal{O}_A, \alpha)$  has a unique KMS state, which occurs at inverse temperature equal to the logarithm  $\ln \rho(A)$  of the spectral radius of  $A$ . Exel and Laca [8] extended this result to Cuntz–Krieger algebras of infinite matrices and also described the KMS states of their Toeplitz extensions. More recently, an Huef, Laca, Raeburn and Sims extended these results to the graph algebras of finite graphs [12] and  $C^*$ -algebras associated to higher-rank graphs [13]. In each case, the Toeplitz extension has many more KMS states than the Cuntz–Krieger algebra, and encodes more information about the underlying object.

In [19], Kribs and Solel studied  $C^*$ -algebras generated by periodic weighted-shift operators on the path spaces of directed graphs. They showed that the  $C^*$ -algebra generated by all such operators can be realised as a direct limit of graph algebras. Specifically, given  $n > 0$ , they construct a graph  $E(n)$  with vertex set  $E^{<n}$ , the space of paths in  $E$  of length at most  $n - 1$ , and they exhibited inclusions  $\mathcal{TC}^*(E(n)) \hookrightarrow \mathcal{TC}^*(E(mn))$ . Upon restriction to the canonical abelian subalgebra in  $\mathcal{TC}^*(E(mn))$ , these inclusions are compatible with a natural surjection  $E^{<mn} \rightarrow E^{<n}$ , so  $\varinjlim \mathcal{TC}^*(E(n))$  has an abelian subalgebra isomorphic to  $C_0(\varprojlim E^{<n})$ . This construction has recently been used to calculate the nuclear dimension of graph algebras and Kirchberg algebras [26, 27].

Here we study the structure of the Kribs–Solel algebras and their Toeplitz extensions, and calculate the KMS states of the associated dynamics. We start in Section 3 by giving a universal description of the Kribs–Solel algebra  $C^*(E, \omega)$  of a directed graph  $E$  corresponding to a sequence  $\omega = (n_k)_{k=1}^\infty$  of positive integers as generated by a copy of  $C^*(E)$

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and a copy of  $C_0(\varprojlim E^{<n_k})$ . We give an analogous description of the Toeplitz extension  $\mathcal{T}(E, \omega)$ . Our approach clarifies the structure of these algebras, and in particular makes transparent the fact that  $C^*(E, \omega)$  and  $\mathcal{T}(E, \omega)$  depend only on  $E$  and the supernatural number determined by  $\omega$  (see Proposition 3.11).

Kribs and Solel use a topological graph  $E(\infty)$  in the sense of Katsura to study some properties of their direct-limit algebras. They show that  $C^*(E, \omega)$  is isomorphic to the topological-graph  $C^*$ -algebra  $C^*(E(\infty))$ , allowing them to plug into Katsura's powerful structure theory. In Section 4 we provide a slightly different description of  $E(\infty)$  that we feel clarifies the construction somewhat, and study its structure in greater depth than appears in [19]. In particular, when  $E$  is finite and strongly connected, we show how to decompose  $E(\infty)$  into irreducible components using Perron-Frobenius theory for the matrices  $A_E^{n_k}$ .

In Section 5 we prove uniqueness theorems for  $C^*(E, \omega)$  and  $\mathcal{T}(E, \omega)$ . The uniqueness theorem for  $\mathcal{T}(E, \omega)$  (Proposition 5.1) is analogous to that for the Toeplitz extension of a graph algebra, and we prove it using that technology. Interestingly, our Cuntz–Krieger uniqueness theorem (Theorem 5.2) for  $C^*(E, \omega)$  requires no aperiodicity hypothesis, emphasising Kribs and Solel's view of these algebras as generalised Bunce–Deddens algebras. We obtain this result by combining Katsura's uniqueness theorem for topological graph  $C^*$ -algebras with Kribs and Solel's observation that their topological graph  $E(\infty)$  has no loops. This also leads to a very satisfactory characterisation of ideal-structure for  $C^*(E, \omega)$  for finite, strongly connected  $E$ :  $C^*(E, \omega)$  decomposes as a direct sum of simple subalgebras indexed by the finite group of integers modulo the greatest common divisor of the supernatural number  $\omega$  and the period of the graph  $E$  in the sense of Perron–Frobenius theory. In particular,  $C^*(E, \omega)$  is simple if and only if  $\omega$  is coprime to the period of  $E$ .

In Section 6, we focus on finite strongly connected graphs  $E$ , and study the KMS states for the gauge-dynamics on  $\mathcal{T}(E, \omega)$ , paying attention to those which factor through  $C^*(E, \omega)$ . Our analysis follows the broad lines of [8, 22], but the inverse-limit structure of the spectrum of the diagonal in  $\mathcal{T}(E, \omega)$  introduces some interesting wrinkles. We reinterpret the KMS condition for states on  $\mathcal{T}(E, \omega)$  as a subinvariance condition for an operator on the space of signed Borel measures on  $\varprojlim E^{<n_k}$  (Theorem 6.10). To construct KMS states on the Toeplitz algebra of a graph  $E$ , one makes use of the path-space representation on  $\ell^2(E^*)$ . It is not *a priori* clear how to construct a corresponding representation of  $\mathcal{T}(E, \omega)$  from the Kribs–Solel approach, but our universal description of  $\mathcal{T}(E, \omega)$  suggests a solution. We use this representation to construct  $\text{KMS}_\beta$  states for all  $\beta > \ln \rho(A_E)$  (Proposition 6.13), and show that there is an affine isomorphism between the  $\text{KMS}_\beta$  simplex of  $\mathcal{T}(E, \omega)$  and the simplex of probability measures on  $\varprojlim E^{<n_k}$  (Corollary 6.15).

Finally, we investigate which KMS states factor through  $C^*(E, \omega)$ . In contrast with [6, 12], strong connectedness of  $E$  is not sufficient to ensure that  $C^*(E, \omega)$  admits a unique KMS state; rather, the extremal KMS states of  $C^*(E, \omega)$  correspond precisely to the direct summands described in Section 5. Following the approach of [15], we describe a formula which always determines a  $\text{KMS}_{\ln \rho(A_E)}$  state  $\phi$  of  $C^*(E, \omega)$ . Restricting this state to each direct summand of  $C^*(E, \omega)$  and normalising yields a family of KMS states all at the same inverse temperature, and we use the results of [6, 14] to show that there cannot be any KMS states for  $C^*(E, \omega)$  at any other temperatures. We prove that these states are precisely the extremal KMS states of  $C^*(E, \omega)$ . We deduce that  $\phi$  is the only KMS state

of  $C^*(E, \omega)$  if and only if  $\omega$  is coprime with the period of  $E$ , and hence if and only if  $C^*(E)$  is simple; we further show that this is equivalent to  $\phi$  being a factor state.

## 2. BACKGROUND

**2.1. Directed graphs and their  $C^*$ -algebras.** We use the convention for graph  $C^*$ -algebras appearing in Raeburn's book [25]. So if  $E = (E^0, E^1, r, s)$  is a directed graph, then a path in  $E$  is a word  $\mu = e_1 \dots e_n$  in  $E^1$  such that  $s(e_i) = r(e_{i+1})$  for all  $i$ , and we write  $r(\mu) = r(e_1)$ ,  $s(\mu) = s(e_n)$ , and  $|\mu| = n$ . As usual, we denote by  $E^*$  the collection of paths of finite length, and  $E^n := \{\mu \in E^* : |\mu| = n\}$ ; we also write  $E^{<n} := \{\mu \in E^* : |\mu| < n\}$ . We borrow the convention from the higher-rank graph literature in which we write, for example  $vE^*$  for  $\{\mu \in E^* : r(\mu) = v\}$ , and  $vE^1w$  for  $\{e \in E^1 : r(e) = v \text{ and } s(e) = w\}$ . The adjacency matrix of  $E$  is then the  $E^0 \times E^0$  integer matrix with  $A_E(v, w) = |vE^1w|$ .

We say that  $E$  is *row-finite* if  $vE^1$  is finite for all  $v \in E^0$ , and that  $E$  has no sources if each  $vE^1$  is nonempty.

If  $E$  is row-finite and has no sources, then a Toeplitz–Cuntz–Krieger  $E$ -family in a  $C^*$ -algebra  $A$  is a pair  $(s, p)$ , where  $s = \{s_e : e \in E^1\} \subseteq A$  is a collection of partial isometries and  $\{p_v : v \in E^0\} \subseteq A$  is a set of mutually orthogonal projections such that  $s_e^* s_e = p_{s(e)}$  for all  $e \in E^1$ , and  $p_v \geq \sum_{e \in vE^1} s_e s_e^*$  for all  $v \in E^0$ . If equality holds in the second relation (for all  $v$ ), then  $(s, p)$  is a Cuntz–Krieger  $E$ -family.

The Toeplitz algebra  $\mathcal{TC}^*(E)$  is the universal  $C^*$ -algebra generated by a Toeplitz–Cuntz–Krieger family ([9]) and the graph algebra  $C^*(E)$  is the universal  $C^*$ -algebra generated by a Cuntz–Krieger  $E$ -family [25, Proposition 1.21].

Kribs and Solel describe their generalised Bunce–Deddens algebras as direct limits of graph  $C^*$ -algebras obtained from the following construction [19, Theorem 4.2].

Let  $E = (E^0, E^1, r, s)$  be a row-finite directed graph with no sources, and fix  $n \in \mathbb{N} \setminus \{0\}$ . Define

$$\begin{aligned} E(n)^0 &:= E^{<n}, & E(n)^1 &:= \{(e, \mu) : e \in E^1, \mu \in s(e)E^{<n}\} \\ s_n(e, \mu) &:= \mu & \text{and} & & r_n(e, \mu) &= \begin{cases} e\mu & \text{if } |\mu| < n-1 \\ r(e) & \text{if } |\mu| = n-1. \end{cases} \end{aligned}$$

Then  $E(n) = (E(n)^0, E(n)^1, r_n, s_n)$  is a row-finite directed graph with no sources. For  $\mu \in E^*$ , we write  $[\mu]_n$  for the unique element of  $E^{<n}$  such that  $\mu = [\mu]_n \mu'$  for some  $\mu'$  with  $|\mu'| \in n\mathbb{N}$ ; we think of  $[\mu]_n$  as the residue of  $\mu$  modulo  $n$ .

It is easy to check that there is a bijection from  $\{(\mu, \nu) : \mu \in E^*, \nu \in s(\mu)E^{<n}\}$  to  $E(n)^*$  that carries  $(\mu, \nu)$  to  $(\mu_1, [\mu_2 \dots \mu_{|\mu|} \nu]_n)(\mu_2, [\mu_3 \dots \mu_{|\mu|} \nu]_n) \dots (\mu_{|\mu|}, \nu)$ . We frequently use this bijection to identify  $E(n)^*$  with  $\{(\mu, \nu) : \mu \in E^*, \nu \in s(\mu)E^{<n}\}$ , and we then have  $s_n(\mu, \nu) = \nu$ , and  $r_n(\mu, \nu) = [\mu\nu]_n$ . This implies, in particular, that the lengths of the paths  $r_n(\mu, \nu)$  and  $s_n(\mu, \nu)$  in  $E^{<n}$  differ by  $|\mu|$  modulo  $n$ . Thus, for  $v, w \in E^0 \subseteq E^{<n}$ , we have

$$(2.1) \quad vE(n)^*w \neq \emptyset \quad \text{if and only if} \quad vE^{jn}w \neq \emptyset \text{ for some } j \in \mathbb{N}.$$

**2.2. The KMS condition.** We use the definition of KMS states given in [2, Definition 5.3.1]. Let  $(A, \mathbb{R}, \alpha)$  be a  $C^*$ -dynamical system. An element  $a \in A$  is analytic for  $\alpha$  if  $t \mapsto \alpha_t(a)$  extends to an entire function  $z \mapsto \alpha_z(a)$  on  $\mathbb{C}$ . Let  $A_\alpha$  denote the collection of

analytic elements of  $A$ . A state  $\phi$  of  $A$  is said to be a KMS state at inverse temperature  $\beta \in \mathbb{R} \setminus \{0\}$  if

$$\phi(ab) = \phi(b\alpha_{i\beta}(a)) \quad \text{for all } a, b \in A_\alpha.$$

It suffices to verify this KMS condition on any  $\alpha$ -invariant set of analytic elements spanning a dense subspace of  $A$ . Proposition 5.3.3 of [2] says that if  $\phi$  is  $\text{KMS}_\beta$  for  $\alpha$ , then  $\phi$  is  $\alpha$ -invariant. If  $\beta = 0$ , then the KMS condition above reduces to requiring that  $\phi$  is a trace, and we then impose  $\alpha$ -invariance as an additional requirement.

**2.3. The Perron–Frobenius theorem.** Let  $X$  be a finite set. A matrix  $A \in M_X(\mathbb{C})$  is *irreducible* if, for all  $x, y \in X$ , there exists  $n \in \mathbb{N}$  such that  $A^n(x, y) \neq 0$ . We say that a matrix is *nonnegative* if all of its entries are nonnegative.

Let  $A$  be an irreducible nonnegative matrix. The Perron–Frobenius theorem (see, for example, [28, Theorem 1.5]) says that the spectral radius  $\rho(A)$  is an eigenvalue of  $A$  with a positive eigenvector, and that  $\rho(A)$  is a simple root of the characteristic polynomial of  $A$ . We call the unique positive eigenvector with eigenvalue  $\rho(A)$  and unit 1-norm the *unimodular Perron–Frobenius eigenvector* of  $A$ .

**2.4. The space of finite signed Borel measures.** If  $M$  is a  $\sigma$ -algebra of subsets of a set  $X$ , then a real-valued function  $m$  defined on  $M$  is said to be a *finite signed measure* if  $m(\emptyset) = 0$  and  $m$  is completely additive.

Suppose that  $X$  is a compact Hausdorff space. We denote by  $\mathcal{M}(X)$  the space of all finite signed Borel measures on  $X$ , by  $\mathcal{M}^+(X)$  the subset of  $\mathcal{M}(X)$  consisting of positive Borel measures, and by  $\mathcal{M}_1^+(X)$  the subset of  $\mathcal{M}^+(X)$  consisting of probability measures on  $X$ .

Let  $m \in \mathcal{M}(X)$ . By the Hahn decomposition theorem [1, Theorem 8.2] there are sets  $P, N \subseteq X$  such that  $X = P \cup N$  and  $P \cap N = \emptyset$ , and such that  $m(E \cap P) \geq 0$  and  $m(E \cap N) < 0$  for all Borel  $E \subseteq X$ .

Let  $m^+$  and  $m^-$  be given by  $m^+(E) = m(E \cap P)$  and  $m^-(E) = -m(E \cap N)$  for Borel  $E$ . Then  $m^+, m^- \in \mathcal{M}^+(X)$ . The Jordan decomposition theorem [1, Theorem 8.5] says that  $m = m^+ - m^-$  and that if  $m', m'' \in \mathcal{M}^+(X)$  satisfy  $m = m' - m''$ , then  $m'(E) \geq m^+(E)$  and  $m''(E) \geq m^-(E)$  for all Borel  $E \subseteq X$ .

The space  $\mathcal{M}(X)$  of finite signed measures is a real Banach space under the norm  $\|m\| = m^+(X) + m^-(X)$ .

### 3. THE KRIBS–SOLEL ALGEBRAS AND THEIR TOEPLITZ EXTENSIONS

In this section, we describe an alternative presentation of Kribs and Solel’s  $C^*$ -algebras  $\mathcal{TC}^*(E(n))$  and  $C^*(E(n))$ , and of their direct-limit algebras  $A(n)$  and  $B(n)$ . We show that  $\mathcal{TC}^*(E(n))$  is the universal  $C^*$ -algebra generated by a Toeplitz–Cuntz–Krieger  $E$ -family and mutually orthogonal projections indexed by  $E^{<n}$ . This presentation has the advantage that the connecting maps  $\mathcal{TC}^*(E(n)) \hookrightarrow \mathcal{TC}^*(E(nm))$  have a particularly simple form: they preserve the generating Toeplitz–Cuntz–Krieger family, and resolve the projection associated to each  $\mu \in E^{<n}$  into a sum of projections associated to paths of the form  $\mu\tau \in E^{<nm}$ . This leads to a very natural presentation of  $A(n)$  in terms of a Toeplitz–Cuntz–Krieger  $E$ -family and a representation of the algebra of continuous functions on a natural projective limit of the  $E^{<n}$ . We show that all of this descends naturally to the  $C^*(E(n))$  and  $B(n)$ .

**Definition 3.1.** Let  $E$  be a row-finite directed graph with no sources, and fix  $n \in \mathbb{N} \setminus \{0\}$ . A *Toeplitz  $n$ -representation* of  $E$  in a  $C^*$ -algebra  $A$  is a triple  $(T, Q, \Theta)$  where  $(T, Q)$  is a Toeplitz–Cuntz–Krieger  $E$ -family in  $A$ , and  $\Theta = \{\Theta_\mu : \mu \in E^{<n}\}$  is a collection of mutually orthogonal projections such that  $Q_v = \sum_{\mu \in vE^{<n}} \Theta_\mu$  for all  $v \in E^0$ , and

$$(3.1) \quad T_e^* \Theta_\mu = \begin{cases} \Theta_{\mu'} T_e^* & \text{if } \mu = e\mu' \\ \sum_{e\nu \in E^n} \Theta_\nu T_e^* & \text{if } \mu = r(e) \\ 0 & \text{otherwise.} \end{cases}$$

If  $(T, Q)$  is a Cuntz–Krieger  $E$ -family, we call  $(T, Q, \Theta)$  a *Cuntz–Krieger  $n$ -representation* of  $E$ .

We show that Kribs and Solel’s  $\mathcal{TC}^*(E(n))$  is universal for Toeplitz  $n$ -representations of  $E$  and that  $C^*(E(n))$  is universal for Cuntz–Krieger  $n$ -representations. We first describe a convenient family of spanning elements. We will need the following notation: given a directed graph  $E$ ,  $n > 0$  and  $\mu \in E^*$ , we write  $\tau_n(\mu)$  for the unique element of  $E^{<n}$  such that  $\mu = \mu' \tau_n(\mu)$  with  $|\mu'| \in n\mathbb{N}$ ; so  $|\tau_n(\mu)| \equiv |\mu| \pmod{n}$ , and  $\mu = \mu' \tau_n(\mu)$ .

**Lemma 3.2.** *Let  $E$  be a row-finite directed graph with no sources, and take  $n > 0$ . Let  $(T, Q, \Theta)$  be a Toeplitz  $n$ -representation of  $E$ , and fix  $\mu \in E^*$  and  $\alpha \in E^{<n}$ .*

- (1) *If  $|\mu| \in n\mathbb{N}$ , then  $T_\mu^* \Theta_{r(\mu)} = \Theta_{s(\mu)} T_\mu^*$ .*
- (2)

$$T_\mu^* \Theta_\alpha = \begin{cases} \Theta_{\alpha'} T_\mu^* & \text{if } \alpha = \mu\alpha' \\ \Theta_{s(\mu)} T_\mu^* & \text{if } \mu = \alpha\mu' \text{ and } |\mu'| \in n\mathbb{N} \\ \sum_{|\tau_n(\mu')\lambda|=n} \Theta_\lambda T_\mu^* & \text{if } \mu = \alpha\mu' \text{ and } |\mu'| \notin n\mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* (1) We calculate

$$(3.2) \quad \begin{aligned} T_\mu^* \Theta_{r(\mu)} &= T_{\mu_{|\mu|}}^* \cdots T_{\mu_2}^* T_{\mu_1}^* \Theta_{r(\mu_1)} = T_{\mu_{|\mu|}}^* \cdots T_{\mu_2}^* \left( \sum_{\mu_1 \lambda \in E^n} \Theta_\lambda T_{\mu_1}^* \right) \\ &= T_{\mu_{|\mu|}}^* \cdots T_{\mu_3}^* \left( \sum_{\mu_1 \mu_2 \lambda \in E^n} \Theta_\lambda T_{\mu_1 \mu_2}^* \right) = \cdots = \Theta_{s(\mu)} T_\mu^*. \end{aligned}$$

- (2) First suppose that  $\alpha = \mu\alpha'$ . Then

$$T_\mu^* \Theta_\alpha = T_{\mu_{|\mu|}}^* \cdots T_{\mu_1}^* \Theta_\alpha = T_{\mu_{|\mu|}}^* \cdots T_{\mu_2}^* \Theta_{\alpha_2 \cdots \alpha_n} T_{\mu_1}^* = \cdots = \Theta_{\alpha'} T_\mu^*.$$

Now suppose that  $\mu = \alpha\mu'$ . Write  $\mu' = \mu'' \tau_n(\mu')$ . Then  $|\mu''| \in n\mathbb{N}$ , so we calculate, using part (1) at the fourth equality,

$$T_\mu^* \Theta_\alpha = T_{\mu'}^* T_\alpha^* \Theta_\alpha = T_{\mu'}^* \Theta_{s(\alpha)} T_\alpha^* = T_{\tau_n(\mu')}^* T_{\mu'}^* \Theta_{r(\mu'')} T_\alpha^* = T_{\tau_n(\mu')}^* \Theta_{s(\mu'')} T_{\alpha\mu''}^*.$$

If  $|\mu'| \in n\mathbb{N}$ , then  $\alpha\mu'' = \mu$  and  $\tau_n(\mu') = s(\mu)$ , so the preceding displayed equation gives  $T_\mu^* \Theta_\alpha = \Theta_{s(\mu)} T_\mu^*$ . Otherwise, we repeat the first  $|\mu''|$  steps of the calculation (3.2) to obtain

$$T_\mu^* \Theta_\alpha = \sum_{|\tau_n(\mu')\lambda|=n} \Theta_\lambda T_\mu^*.$$

Finally, if  $\mu \neq \alpha\mu'$  and  $\alpha \neq \mu\alpha'$ , then we can write  $\mu = \lambda e\mu'$  and  $\alpha = \lambda f\alpha'$  for distinct  $e, f \in E^1$ . Using the first case in part (2), we obtain

$$T_\mu^* \Theta_\alpha = T_{\mu'}^* T_e^* \Theta_{f\alpha'} T_\lambda^*,$$

which is zero by the displayed relation in Definition 3.1.  $\square$

**Lemma 3.3.** *Let  $E$  be a row-finite directed graph with no sources, take  $n > 0$  and suppose that  $(T, Q, \Theta)$  is a Toeplitz  $n$ -representation of  $E$ . For  $\alpha, \beta, \gamma, \delta \in E^*$  and  $\mu, \nu \in E^{<n}$ ,*

$$(T_\alpha \Theta_\mu T_\beta^*)(T_\gamma \Theta_\nu T_\delta^*) = \begin{cases} T_\alpha \Theta_\mu T_{\delta\beta'}^* & \text{if } \beta = \gamma\beta' \text{ and } \nu = \beta'\mu \\ T_\alpha \Theta_\mu T_{\delta\nu\rho}^* & \text{if } \beta = \gamma\nu\rho \text{ with } |\rho\mu| \in n\mathbb{N} \\ T_{\alpha\gamma'} \Theta_\nu T_\delta^* & \text{if } \gamma = \beta\gamma' \text{ and } \mu = \gamma'\nu \\ T_{\alpha\mu\rho} \Theta_\nu T_\delta^* & \text{if } \gamma = \beta\mu\rho \text{ with } |\rho\nu| \in n\mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We consider the case where  $|\beta| \geq |\gamma|$ ; the case where  $|\gamma| > |\beta|$  will then follow by taking adjoints. By [25, Corollary 1.14(b)], we have

$$(T_\alpha \Theta_\mu T_\beta^*)(T_\gamma \Theta_\nu T_\delta^*) = \begin{cases} T_\alpha \Theta_\mu T_{\beta'}^* \Theta_\nu T_\delta^* & \text{if } \beta = \gamma\beta' \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that  $\beta = \gamma\beta'$ . By Lemma 3.2(2) we have

$$\begin{aligned} (T_\alpha \Theta_\mu T_\beta^*)(T_\gamma \Theta_\nu T_\delta^*) &= \begin{cases} T_\alpha \Theta_\mu \Theta_{\nu'} T_{\delta\beta'}^* & \text{if } \nu = \beta'\nu' \\ T_\alpha \Theta_\mu \Theta_{s(\beta')} T_{\delta\beta'}^* & \text{if } \beta' = \nu\rho \text{ with } |\rho| \in n\mathbb{N} \\ T_\alpha \Theta_\mu \sum_{\tau_n(\rho)\lambda \in E^n} \Theta_\lambda T_{\delta\beta'}^* & \text{if } \beta' = \nu\rho \text{ with } |\rho| \notin n\mathbb{N} \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} T_\alpha \Theta_\mu T_{\delta\beta'}^* & \text{if } \nu = \beta'\mu \\ & \text{or } \beta' = \nu\rho \text{ with } |\rho| \in n\mathbb{N} \text{ and } \mu = s(\beta) \\ & \text{or } \beta' = \nu\rho \text{ and } \tau_n(\rho)\mu \in E^n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since  $\tau_n(\rho)\mu \in E^n$  if and only if  $|\rho\mu| \in n\mathbb{N}$ , the result follows.  $\square$

**Theorem 3.4.** *Let  $E$  be a row-finite directed graph with no sources, and let  $(t_{(e,\mu)}, q_\mu)$  be the universal Toeplitz–Cuntz–Krieger  $E(n)$ -family in  $\mathcal{TC}^*(E(n))$ . Then the elements*

$$t_{n,e} := \sum_{(e,\mu) \in E(n)^1} t_{(e,\mu)}, \quad q_{n,v} := \sum_{\mu \in vE^{<n}} q_\mu, \quad \text{and} \quad \theta_{n,\mu} := q_\mu$$

*constitute a Toeplitz  $n$ -representation of  $E$  and generate  $\mathcal{TC}^*(E(n))$ . For every Toeplitz  $n$ -representation  $(T, Q, \Theta)$  of  $E$  in a  $C^*$ -algebra  $B$ , there is a  $C^*$ -homomorphism  $\pi_{T,Q,\Theta} : \mathcal{TC}^*(E(n)) \rightarrow B$  such that  $\pi_{T,Q,\Theta}(t_{n,e}) = T_e$ ,  $\pi_{T,Q,\Theta}(q_{n,v}) = Q_v$  and  $\pi_{T,Q,\Theta}(\theta_{n,\mu}) = \Theta_\mu$ .*

*If  $(T, Q)$  is a Cuntz–Krieger  $E$ -family, then  $\pi_{T,Q,\Theta}$  factors through a homomorphism  $\tilde{\pi}_{T,Q,\Theta} : C^*(E(n)) \rightarrow B$ .*

*Proof.* Routine calculations show that  $(t, q, \theta)$  is a Toeplitz  $n$ -representation of  $E$ . We have  $t_{(e,\mu)} = t_{n,e} \theta_{n,\mu}$  for each  $(e, \mu) \in E(n)^1$  and  $q_\mu = \theta_{n,\mu}$  for each  $\mu \in E(n)^0$ , and so the  $t_{n,e}$ , the  $q_{n,v}$  and the  $\theta_{n,\mu}$  generate  $\mathcal{TC}^*(E(n))$ .

Fix a Toeplitz  $n$ -representation  $(T, Q, \Theta)$ . Routine calculations show that the elements  $T_{(e,\mu)} := T_e \Theta_\mu$  and  $Q_\mu := \Theta_\mu$  form a Toeplitz–Cuntz–Krieger  $E(n)$  family, and so induce the desired homomorphism  $\pi_{T,Q,\Theta}$ . For each  $v \in E^0$ , we have

$$\sum_{\mu \in vE^{<n}} \sum_{(e,\nu) \in \mu E(n)^1} T_{(e,\nu)} T_{(e,\nu)}^* = \sum_{e \in vE^1} \sum_{(e,\nu) \in E(n)^1} T_{(e,\nu)} T_{(e,\nu)}^* = \sum_{e \in vE^1} T_e T_e^*.$$

So if  $(T, Q)$  is a Cuntz–Krieger  $E$ -family, then the  $T_{(e,\mu)}$  and  $Q_\mu$  form a Cuntz–Krieger  $E(n)$  family and so  $\pi_{T,Q,\Theta}$  factors through  $\tilde{\pi}_{T,Q,\Theta} : C^*(E(n)) \rightarrow B$ .  $\square$

**Notation 3.5.** Using Theorem 3.4, we write  $\mathcal{T}(E, n)$  for  $\mathcal{T}C^*(E(n))$  and regard it as the universal  $C^*$ -algebra generated by a Toeplitz  $n$ -representation  $(t_{n,e}, q_{n,v}, \theta_{n,\mu})$  of  $E$ . We also write  $C^*(E, n)$  for  $C^*(E(n))$ , and regard it as the universal  $C^*$ -algebra generated by a Cuntz–Krieger  $n$ -representation  $(s_{n,e}, p_{n,v}, \varepsilon_{n,\mu})$ .

Next, we describe the homomorphisms  $\mathcal{T}C^*(E(n)) \hookrightarrow \mathcal{T}C^*(E(mn))$  and  $C^*(E(n)) \hookrightarrow C^*(E(mn))$  of Kribs and Solel in terms of the universal properties just described.

**Proposition 3.6.** *Let  $E$  be a row-finite directed graph with no sources. Take integers  $m, n \geq 1$ . There is an injective homomorphism  $i_{n,mn} : \mathcal{T}(E, n) \rightarrow \mathcal{T}(E, mn)$  such that*

$$i_{n,mn}(t_{n,e}) = t_{mn,e}, \quad i_{n,mn}(q_{n,v}) = q_{mn,v}, \quad \text{and} \quad i_{n,mn}(\theta_{n,\mu}) = \sum_{\nu \in E^{<mn}, [\nu]_n = \mu} \theta_{mn,\nu}.$$

Moreover  $i_{n,mn}$  descends to an injection of  $C^*(E, n)$  into  $C^*(E, mn)$ .

*Proof.* For  $e \in E^1$ ,  $v \in E^0$  and  $\mu \in E^{<n}$ , define  $T_e := t_{mn,e}$ ,  $Q_v := q_{mn,v}$  and  $\Theta_\mu = \sum_{\nu \in E^{<mn}, [\nu]_n = \mu} \theta_{mn,\nu}$ . Straightforward calculations show that  $(T, Q, \Theta)$  is a Toeplitz  $n$ -representation of  $E$ , so the universal property of  $\mathcal{T}(E, n)$  gives a homomorphism  $i_{n,mn}$  satisfying the desired formulas. Using the formulas for the generators of  $\mathcal{T}(E, n)$  in Theorem 3.4, we see that for  $\mu \in E^{<n}$ ,

$$i_{n,mn}\left(q_\mu - \sum_{(e,\tau) \in \mu E(n)^1} t_{(e,\tau)} t_{(e,\tau)}^*\right) = \sum_{\nu \in E^{<mn}, [\nu]_n = \mu} \left(q_\nu - \sum_{(e,\tau) \in \nu E(mn)^1} t_{(e,\tau)} t_{(e,\tau)}^*\right).$$

Theorem 4.1 of [9] implies that each term on the right hand side of the preceding displayed equation is nonzero, and then also that  $i_{n,mn}$  is injective. Hence  $i_{n,mn}$  is also injective.

For the final statement, observe that  $i_{n,mn}$  clearly preserves the Cuntz–Krieger relation, so it descends to a homomorphism  $\tilde{i}_{n,mn} : C^*(E, n) \rightarrow C^*(E, mn)$ . A routine application of the gauge-invariant uniqueness theorem [3, Theorem 2.1] for  $C^*(E(n))$  shows that  $\tilde{i}_{n,mn}$  is injective.  $\square$

Using the homomorphisms of the preceding proposition, we can form the direct limits  $\varinjlim \mathcal{T}(E, n_k)$  and  $\varinjlim C^*(E, n_k)$ . We write  $i_{n_k, n_l} : \mathcal{T}(E, n_k) \rightarrow \mathcal{T}(E, n_l)$  for the connecting homomorphism with  $k < l$ , and we write  $i_{n_k, \infty} : \mathcal{T}(E, n_k) \rightarrow \varinjlim \mathcal{T}(E, n_k)$  for the canonical inclusion. We will also use these same symbols to denote the corresponding maps in the direct system associated to the  $C^*(E, n_k)$ .

Fix a directed graph  $E$ . For  $m, n \in \mathbb{N} \setminus \{0\}$  such that  $m \mid n$ , we define  $p_{n,m} : E^{<n} \rightarrow E^{<m}$  by  $p_{n,m}(\nu) = [\nu]_m$ . Consider a sequence  $(n_k)_{k=1}^\infty$  such that  $n_k \mid n_{k+1}$  for all  $k$ . The projective limit  $(\varprojlim E^{<n_k}, p_{n_{k+1}, n_k})$  can be realised as the topological subspace

$$\left\{ (\mu_k)_{k=1}^\infty \in \prod_{k=1}^\infty E^{<n_k} : \mu_k = [\mu_{k+1}]_{n_k} \text{ for all } k \in \mathbb{N} \right\}.$$

For a sequence  $(n_k)_{k=1}^\infty$  as above and a directed graph  $E$ , given  $k \in \mathbb{N}$  and  $\mu \in E^{<n_k}$ , we write  $Z(\mu, k)$  for the cylinder set  $\{(\nu_i)_{i=1}^\infty \in \varprojlim E^{<n_k} : \nu_k = \mu\}$ . Observe that the  $Z(\mu, k)$  are the canonical compact open basis sets for the projective limit space regarded as a subspace of the infinite product  $\prod_{k=1}^\infty E^{<n_k}$ . We write  $\chi_{Z(\mu, k)}$  for the characteristic function of  $Z(\mu, k) \subseteq \varprojlim E^{<n_k}$ .

**Definition 3.7.** Let  $E$  be a row-finite directed graph with no sources, and suppose that  $\omega = (n_k)_{k=1}^\infty$  is a sequence of nonzero natural numbers such that  $n_k \mid n_{k+1}$  for all  $k$ . A *Toeplitz  $\omega$ -representation of  $E$*  is a triple  $(T, Q, \psi)$  consisting of a Toeplitz–Cuntz–Krieger  $E$ -family in a  $C^*$ -algebra  $B$  and a homomorphism  $\psi : C_0(\varprojlim E^{<n_k}) \rightarrow B$  such that  $Q_w = \psi(\chi_{Z(w, 1)})$  for all  $w \in E^0$ , and

$$T_e^* \psi(\chi_{Z(\mu, k)}) = \begin{cases} \psi(\chi_{Z(\mu', k)}) T_e^* & \text{if } \mu = e\mu' \\ \sum_{e\lambda \in E^{n_k}} \psi(\chi_{Z(\lambda, k)}) T_e^* & \text{if } \mu = r(e) \\ 0 & \text{otherwise} \end{cases}$$

for all  $e \in E^1$ ,  $k \in \mathbb{N}$  and  $\mu \in E^{<n_k}$ . If the pair  $(T, Q)$  is a Cuntz–Krieger  $E$ -family, then we call  $(T, Q, \psi)$  a *Cuntz–Krieger  $\omega$ -representation*, or just an  $\omega$ -representation of  $E$ .

We show that the universal  $C^*$ -algebra generated by an  $\omega$ -representation coincides with Kribs and Solel’s algebra  $\varinjlim C^*(E(n_k))$ . We first need a multiplication formula analogous to that of Lemma 3.3. To lighten notation a bit, given a homomorphism  $\psi : C_0(\varprojlim E^{<n_k}) \rightarrow B$ , we will write  $\psi_{(\mu, k)}$  for the image of  $\chi_{Z(\mu, k)}$  under  $\psi$ , which is a projection in  $B$ .

**Lemma 3.8.** Let  $E$  be a row-finite directed graph with no sources, and let  $\omega = (n_k)_{k=1}^\infty$  be a sequence of nonzero natural numbers such that  $n_k \mid n_{k+1}$  for all  $k$ . Let  $(T, Q, \psi)$  be a Toeplitz  $\omega$ -representation of  $E$ . For  $\alpha, \beta, \gamma, \delta \in E^*$ ,  $k \geq 1$  and  $\mu, \nu \in E^{<n_k}$ , we have

$$(T_\alpha \psi_{(\mu, k)} T_\beta^*) (T_\gamma \psi_{(\nu, k)} T_\delta^*) = \begin{cases} T_\alpha \psi_{(\mu, k)} T_{\delta\beta'}^* & \text{if } \beta = \gamma\beta' \text{ and } \nu = \beta'\mu \\ T_\alpha \psi_{(\mu, k)} T_{\delta\nu\rho}^* & \text{if } \beta = \gamma\nu\rho \text{ with } |\rho\mu| \in n\mathbb{N} \\ T_{\alpha\gamma'} \psi_{(\nu, k)} T_\delta^* & \text{if } \gamma = \beta\gamma' \text{ and } \mu = \gamma'\nu \\ T_{\alpha\mu\rho} \psi_{(\nu, k)} T_\delta^* & \text{if } \gamma = \beta\mu\rho \text{ with } |\rho\nu| \in n\mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

In particular,  $C^*(T, Q, \psi) = \overline{\text{span}}\{T_\alpha \psi_{(\mu, k)} T_\beta^* : k \geq 1, \mu \in E^{<n_k}, \alpha, \beta \in E^* r(\mu)\}$ .

*Proof.* The first statement follows from the observation that each  $(T, Q, \psi_{(\cdot, k)})$  is a Toeplitz  $n_k$ -representation, and Lemma 3.3. For the second statement, first observe that the set on the right-hand side contains each  $T_\alpha = \sum_{\mu \in s(\alpha) E^{<n_1}} T_\alpha \psi_{(\mu, 1)} T_{s(\alpha)}^*$ , each  $Q_v = \sum_{\mu \in v E^{<n_1}} T_v \psi_{(\mu, 1)} T_v^*$  and each  $\psi_{(\mu, k)} = T_{r(\mu)} \psi_{(\mu, k)} T_{r(\mu)}^*$ . It is clearly closed under adjoints. So it suffices to show that it is closed under multiplication. To see this, we consider a product  $T_\alpha \psi_{(\mu, k)} T_\beta^* T_\gamma \psi_{(\nu, l)} T_\delta^*$ . Suppose that  $k \geq l$  (the case where  $k < l$  will follow by taking adjoints). Then  $Z(\nu, l) = \bigsqcup_{\lambda \in E^{<n_k}, [\lambda]_{n_l} = \nu} Z(\lambda, k)$ , and so we have

$$T_\alpha \psi_{(\mu, k)} T_\beta^* T_\gamma \psi_{(\nu, l)} T_\delta^* = \sum_{\lambda \in E^{<n_k}, [\lambda]_{n_l} = \nu} T_\alpha \psi_{(\mu, k)} T_\beta^* T_\gamma \psi_{(\lambda, k)} T_\delta^*,$$

and this belongs to  $\overline{\text{span}}\{T_\alpha \psi_{(\mu, k)} T_\beta^* : k \geq 1, \mu \in E^{<n_k}, \alpha, \beta \in E^* r(\mu)\}$  by the first statement.  $\square$



**Theorem 3.9.** *Let  $E$  be a row-finite directed graph with no sources, and let  $\omega = (n_k)_{k=1}^\infty$  be a sequence of nonzero natural numbers such that  $n_k \mid n_{k+1}$  for all  $k$ . There is a Toeplitz  $\omega$ -representation  $(t, q, \pi)$  of  $E$  in  $\varinjlim \mathcal{T}(E, n_k)$  such that*

$$t_e = i_{n_1, \infty}(t_{n_1, e}), \quad q_v = i_{n_1, \infty}(q_{n_1, v}), \quad \text{and} \quad \pi_{(\mu, k)} = i_{n_k, \infty}(\theta_{n_k, \mu})$$

for all  $e \in E^1$ , all  $v \in E^0$ , and all  $k \in \mathbb{N}$  and  $\mu \in E^{<n_k}$ . This Toeplitz  $\omega$ -representation is universal in the sense that if  $(T, Q, \psi)$  is a Toeplitz  $\omega$ -representation of  $E$  in a  $C^*$ -algebra  $B$ , then there is a homomorphism  $\varphi_{T, Q, \psi} : \varinjlim \mathcal{T}(E, n_k) \rightarrow B$  such that

$$\varphi_{T, Q, \psi}(t_e) = T_e, \quad \varphi_{T, Q, \psi}(q_v) = Q_v, \quad \text{and} \quad \varphi_{T, Q, \psi} \circ \pi = \psi.$$

*Proof.* We assume without loss of generality that  $n_1 = 1$ . The collection  $(t_{n_1}, q_{n_1})$  is a Toeplitz–Cuntz–Krieger  $E$ -family and since  $i_{n_1, \infty}$  is a homomorphism, it follows that  $t_e := i_{n_1, \infty}(t_{n_1, e})$  and  $q_v = i_{n_1, \infty}(q_{n_1, v})$  is a Toeplitz–Cuntz–Krieger  $E$ -family in  $\varinjlim \mathcal{T}(E, n_k)$ . For each  $k$ , the formula

$$(3.3) \quad \pi_k(\chi_{Z(\mu, k)}) := i_{n_k, \infty}(\theta_{n_k, \mu})$$

gives a homomorphism  $\pi_k : \text{span}\{\chi_{Z(\mu, k)} : \mu \in E^{<n_k}\} \rightarrow \varinjlim \mathcal{T}(E, n_k)$ . So the universal property of  $C_0(\varinjlim E^{<n_k}) \cong \varinjlim C_0(E^{<n_k})$  yields a homomorphism  $\pi : C_0(\varinjlim E^{<n_k}) \rightarrow \varinjlim \mathcal{T}(E, n_k)$  satisfying  $\pi_{(\mu, k)} = i_{n_k, \infty}(\theta_{n_k, \mu})$ .

We check that  $(t, q, \pi)$  is a Toeplitz  $\omega$ -representation. Since  $n_1 = 1$ , for  $w \in E^0$ , we have  $q_w = i_{n_1, \infty}(q_{n_1, w}) = i_{n_1, \infty}(\theta_{n_1, w}) = \pi_{Z(w, 1)}$ . Take  $e \in E^1$  and  $\mu \in E^{<n_k}$ . Then

$$\begin{aligned} t_e^* \pi_{(\mu, k)} &= i_{n_1, \infty}(t_{n_1, e}^*) i_{n_k, \infty}(\theta_{n_k, \mu}) = i_{n_k, \infty}(i_{n_1, n_k}(t_{n_1, e}^*) \theta_{n_k, \mu}) \\ &= i_{n_k, \infty}(t_{n_k, e}^* \theta_{n_k, \mu}) = \begin{cases} i_{n_k, \infty}(\theta_{n_k, \mu'} t_{n_k, e}^*) & \text{if } \mu = e\mu' \\ i_{n_k, \infty}(\sum_{e\lambda \in E^{n_k}} \theta_{n_k, \lambda} t_{n_k, e}^*) & \text{if } \mu = r(e) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \pi_{(\mu', k)} t_e^* & \text{if } \mu = e\mu' \\ \sum_{e\lambda \in E^{n_k}} \pi_{(\lambda, k)} t_e^* & \text{if } \mu = r(e) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

So  $(t, q, \pi)$  is a Toeplitz  $\omega$ -representation of  $E$  in  $\varinjlim \mathcal{T}(E, n_k)$ .

Let  $(T, Q, \psi)$  be another  $\omega$ -representation of  $E$  in  $B$ , and fix  $k \in \mathbb{N}$ . For  $\mu \in E^{<n_k}$  let  $\Theta_\mu := \psi_{(\mu, k)}$ . Quick calculations show that  $(T, Q, \Theta)$  is a Toeplitz  $n_k$ -representation of  $E$ . The universal property of  $\mathcal{T}(E, n_k)$  gives a homomorphism  $\varphi_{n_k, \infty} : \mathcal{T}(E, n_k) \rightarrow B$  satisfying

$$\varphi_{n_k, \infty}(t_e) = T_e, \quad \varphi_{n_k, \infty}(q_v) = Q_v, \quad \text{and} \quad \varphi_{n_k, \infty}(\theta_{n_k, \mu}) = \psi_{(\mu, k)}.$$

We check that  $\varphi_{n_{k+1}, \infty} \circ i_{n_k, n_{k+1}} = \varphi_{n_k, \infty}$ . We have

$$\varphi_{n_{k+1}, \infty} \circ i_{n_k, n_{k+1}}(t_{n_k, e}) = \varphi_{n_{k+1}, \infty}(t_{n_{k+1}, e}) = T_e = \varphi_{n_k, \infty}(t_{n_k, e}),$$

and similarly  $\varphi_{n_{k+1}, \infty} \circ i_{n_k, n_{k+1}}(q_{n_k, v}) = Q_v = \varphi_{n_k, \infty}(q_{n_k, v})$ . For  $\mu \in E^{<n_k}$ ,

$$\begin{aligned} \varphi_{n_{k+1}, \infty}(i_{n_k, n_{k+1}}(\theta_{n_k, \mu})) &= \varphi_{n_k, \infty}\left(\sum_{\lambda \in E^{<n_{k+1}}, [\lambda]_{n_k} = \mu} \theta_{n_{k+1}, \lambda}\right) \\ &= \psi\left(\sum_{\lambda \in E^{<n_{k+1}}, [\lambda]_{n_k} = \mu} \chi_{Z(\lambda, n_{k+1})}\right) = \psi(\chi_{Z(\mu, k)}) = \varphi_{n_k, \infty}(\theta_{n_k, \mu}). \end{aligned}$$

The universal property of  $\varinjlim \mathcal{T}(E, n_k)$  now gives a homomorphism  $\varphi_{T,Q,\psi}$  making the diagram

$$\begin{array}{ccc}
 \mathcal{T}(E, n_k) & \xrightarrow{i_{n_k, n_{k+1}}} & \mathcal{T}(E, n_{k+1}) \\
 \searrow i_{n_k, \infty} & & \swarrow i_{n_{k+1}, \infty} \\
 & \varinjlim \mathcal{T}(E, n_k) & \\
 \swarrow \varphi_{n_k, \infty} & \downarrow \varphi_{T,Q,\psi} & \searrow \varphi_{n_{k+1}, \infty} \\
 & B &
 \end{array}$$

commute, and this homomorphism has the desired properties.  $\square$

Given  $E$  and  $\omega$  as in Theorem 3.9, we write  $\mathcal{T}(E, \omega)$  for the universal  $C^*$ -algebra generated by a Toeplitz  $\omega$ -representation of  $E$ . Since the universal  $C^*$ -algebra for a given set of generators and relations is unique up to canonical isomorphism, we can and will identify  $\mathcal{T}(E, \omega)$  with  $\varinjlim \mathcal{T}(E, n_k)$  via the homomorphism of Theorem 3.9.

The following theorem follows from the same argument as Theorem 3.9.

**Theorem 3.10.** *Let  $E$  be a row-finite directed graph with no sources, and let  $\omega = (n_k)_{k=1}^\infty$  be a sequence of nonzero natural numbers such that  $n_k \mid n_{k+1}$  for all  $k$ . There is an  $\omega$ -representation  $(s, p, \rho)$  of  $E$  in  $\varinjlim C^*(E, n_k)$  such that*

$$s_e = i_{n_1, \infty}(s_{n_1, e}), \quad p_v = i_{n_1, \infty}(p_{n_1, v}), \quad \text{and} \quad \rho_{(\mu, k)} = i_{n_k, \infty}(\varepsilon_{n_k, \mu})$$

for all  $e \in E^1$ , all  $v \in E^0$ , and all  $k \in \mathbb{N}$  and  $\mu \in E^{< n_k}$ . This  $\omega$ -representation is universal in the sense that if  $(S, P, \psi)$  is an  $\omega$ -representation of  $E$  in a  $C^*$ -algebra  $B$ , then there is a homomorphism  $\varphi_{S, P, \psi} : \varinjlim C^*(E, n_k) \rightarrow B$  such that

$$\varphi_{S, P, \psi}(s_e) = S_e, \quad \varphi_{S, P, \psi}(p_v) = P_v, \quad \text{and} \quad \varphi_{S, P, \psi} \circ \rho = \psi.$$

We write  $C^*(E, \omega)$  for the universal  $C^*$ -algebra generated by an  $\omega$ -representation of  $E$ , and we identify it with  $\varinjlim C^*(E, n_k)$  via the homomorphism of the preceding theorem.

Kribs and Solel regard  $\varinjlim C^*(E, n_k)$  as a generalised Bunce–Deddens algebra. Since the Bunce–Deddens algebra  $B_\omega$  is completely determined by the supernatural number  $\omega$ , we expect  $C^*(E, \omega)$  to depend only on  $E$  and the supernatural number associated to  $\omega$ . We give an elementary proof that this is the case using the presentation given in Theorem 3.10. For this, recall that for sequences  $\omega = (n_k)_{k=1}^\infty$  with  $n_k \mid n_{k+1}$  for all  $k$ , and  $\omega' = (m_l)_{l=1}^\infty$  with  $m_l \mid m_{l+1}$  for all  $l$ , we write  $\omega \mid \omega'$  if for every  $k \geq 1$  there exists  $j(k) \geq 1$  such that  $n_k \mid m_{j(k)}$ . The supernatural number  $[\omega]$  associated to  $\omega$  is the collection  $[\omega] := \{\omega' : \omega \mid \omega' \text{ and } \omega' \mid \omega\}$ .

**Proposition 3.11.** *Let  $E$  be a row-finite directed graph with no sources. Let  $\omega = (n_k)_{k=1}^\infty$  and  $\omega' = (m_j)_{j=1}^\infty$  be sequences of nonzero natural numbers such that  $n_k \mid n_{k+1}$  for all  $k$  and  $m_j \mid m_{j+1}$  for all  $j$ . If  $\omega \mid \omega'$ , then there is an injective homomorphism  $\varphi_{\omega, \omega'} : \mathcal{T}(E, \omega) \rightarrow \mathcal{T}(E, \omega')$  such that*

$$(3.4) \quad \varphi_{\omega, \omega'} \circ i_{n_k, \infty} = i_{m_{j(k)}, \infty} \circ i_{n_k, m_{j(k)}} \quad \text{for all } k \geq 1 \text{ and any } j(k) \text{ such that } n_k \mid m_{j(k)}.$$

Moreover,  $\varphi_{\omega, \omega'}$  descends to a homomorphism  $\tilde{\varphi}_{\omega, \omega'} : C^*(E, \omega) \rightarrow C^*(E, \omega')$ . If  $[\omega] = [\omega']$  then  $\varphi_{\omega, \omega'} : \mathcal{T}(E, \omega) \rightarrow \mathcal{T}(E, \omega')$ , and  $\tilde{\varphi}_{\omega, \omega'} : C^*(E, \omega) \rightarrow C^*(E, \omega')$  are isomorphisms.

*Proof.* Fix natural numbers  $j(k)$  such that  $n_k \mid m_{j(k)}$  for all  $k$ . Then  $i_{m_{j(k)},\infty} \circ i_{n_k,m_{j(k)}} : \mathcal{T}(E, n_k) \rightarrow \varinjlim \mathcal{T}(E, m_k)$  is a homomorphism for each  $k$ . Since

$$\begin{aligned} i_{m_{j(k+1)},\infty} \circ i_{n_{k+1},m_{j(k+1)}} \circ i_{n_k,n_{k+1}} &= i_{m_{j(k+1)},\infty} \circ i_{n_k,m_{j(k+1)}} \\ &= i_{m_{j(k+1)},\infty} \circ i_{m_{j(k)},m_{j(k+1)}} \circ i_{n_k,m_{j(k)}} = i_{m_{j(k)},\infty} \circ i_{n_k,m_{j(k)}}, \end{aligned}$$

The universal property of  $\varinjlim \mathcal{T}(E, n_k)$  gives a homomorphism  $\varphi$  that satisfies (3.4). The same argument shows that  $\varphi$  descends to a homomorphism  $\tilde{\varphi} : \varinjlim C^*(E, n_k) \rightarrow \varinjlim C^*(E, m_k)$ . Now suppose that  $\omega' \mid \omega$  as well. The preceding paragraph gives a homomorphism  $\gamma : \mathcal{T}(E, \omega') \rightarrow \mathcal{T}(E, \omega)$  such that  $\gamma \circ i_{m_j,\infty} = i_{n_{k(j)},\infty} \circ i_{m_j,n_{k(j)}}$  for all  $j$ , and which descends to  $\tilde{\gamma} : C^*(E, \omega') \rightarrow C^*(E, \omega)$ . It is routine to check that  $\gamma \circ \phi$  is the identity map on each  $i_{n_k,\infty} \mathcal{T}(E, n_k)$  and symmetrically,  $\phi \circ \gamma$  is the identity on each  $i_{m_j}(\mathcal{T}(E, m_j))$ , so continuity shows that  $\phi$  and  $\gamma$  are mutually inverse; the same argument shows that  $\tilde{\varphi}$  and  $\tilde{\gamma}$  are mutually inverse.  $\square$

#### 4. THE TOPOLOGICAL GRAPH $E(\infty)$

Kribs and Solel construct a topological graph  $E(\infty)$  from a graph  $E$  and a supernatural number  $\omega$ . They show in [19, Theorem 6.3] that  $C^*(E, \omega)$  is isomorphic to the  $C^*$ -algebra  $C^*(E(\infty))$  of this topological graph in the sense of Katsura [16]. Unfortunately, their statement does not give explicit details about the isomorphism, and we shall need these in the sequel. In this section, we give a slightly different description of the topological graph  $E(\infty)$ , and use it to present the details of the isomorphism  $C^*(E, \omega) \cong C^*(E(\infty))$ . For the most part, we are just making explicit some of the details of the proofs of results in [19] and [17], so we keep our presentation short.

First recall that a topological graph  $F$  consists of second-countable locally compact Hausdorff spaces  $F^0$  and  $F^1$  and maps  $r, s : F^1 \rightarrow F^0$  such that  $r$  is continuous and  $s$  is a homeomorphism. Katsura [16] associates to each topological graph  $F$  a  $C^*$ -algebra that we denote  $C^*(F)$ . This  $C^*(F)$  is generated by a homomorphism  $t_F^0 : C_0(F^0) \rightarrow C^*(F)$  and a linear map  $t_F^1 : C_c(F^1) \rightarrow C^*(F)$  satisfying relations reminiscent of the Cuntz–Krieger relations for graph algebras (for a description that avoids the machinery of Hilbert modules, see [23]). The pair  $(t_F^0, t_F^1)$  is called a Cuntz–Krieger  $E$ -pair. When  $F^0$  and  $F^1$  are discrete and countable,  $C^*(F)$  coincides with the usual graph  $C^*$ -algebra described in Section 2.

Now let  $E$  be a row-finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Suppose that  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Let  $X_i = \{w \in E^* : 0 \leq |w| < n_i, |w| \equiv 0 \pmod{n_{i-1}}\}$ , let  $X = \prod_{i=1}^\infty X_i$  and let  $Y = \{y \in X : s(y_k) = r(y_{k+1})\}$ . For each  $e \in E^1$ , let  $D_e = \{y \in Y : r(y_1) = s(e)\}$  and  $R_e = \{y \in Y : \text{for some } l \leq \infty, y_i = r(e) \text{ for all } i < l \text{ and (if } l \neq \infty) y_l = ey' \text{ for some } |y'| \equiv -1 \pmod{n_{l-1}}\}\}$ . For  $y \in D_e$ , write  $i(y)$  for the smallest positive integer such that  $|y_i| < n_i - n_{i-1}$  or  $i(y) = \infty$  if  $|y_i| = n_i - n_{i-1}$  for every  $i$ . If  $i(y) < \infty$ , write  $\sigma_e(y) = u$ , where

$$u_i = \begin{cases} r(e) & \text{if } i < i(y) \\ ey_1 \dots y_{i(y)} & \text{if } i = i(y) \\ y_i & \text{if } i > i(y). \end{cases}$$

If  $i(y) = \infty$ , set  $\sigma_e(y) = (r(e), r(e), \dots)$ .

Kribs and Solel construct a topological graph  $E(\infty)$  with  $E(\infty)^0 = Y$ ,  $E(\infty)^1 = \{(e, y) \in E^1 \times Y : y \in D_e\}$ ,  $s_{E(\infty)}(e, y) = y$  and  $r_{E(\infty)}(e, y) = \sigma_e(y)$ . Here we give another presentation of  $E(\infty)$  which is more natural within our framework.

**Lemma 4.1.** *Let  $E$  be a row-finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Suppose that  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Define  $F_{E,\omega}^0 = \varprojlim E^{<n_k}$ ,  $F_{E,\omega}^1 = \{(e, x) \in E^1 \times \varprojlim E^{<n_k} : r(x_1) = s(e)\}$ ,  $s_F(e, x) = x$ , and  $r_F(e, x)_k = r_{n_k}(e, x_k) = [ex_k]_{n_k}$ . Then  $\phi = (\phi^0, \phi^1) : E(\infty) \rightarrow F$  defined by  $\phi^0(y)_i = y_1 y_2 \dots y_i$  for  $y \in E(\infty)^0$  and  $\phi^1(e, y)_i = (e, \phi^0(y))$  for  $(e, y) \in E(\infty)^1$  is an isomorphism of topological graphs.*

*Proof.* We abbreviate  $F^i := F_{E,\omega}^i$ ,  $i = 0, 1$ . Define  $\psi = (\psi^0, \psi^1) : F \rightarrow Y$  by  $\psi^0(x)_1 = x_1$  and  $[x_{i+1}]_{n_i} \psi^0(x)_{i+1} = x_{i+1}$  for all  $i \geq 1$ , and  $\psi^1(e, x) = (e, \psi^0(x))$ . We have  $\psi^0(\phi^0(y))_i = \psi^0((y_1 \dots y_j)_{j=1}^\infty)_i$ . Since  $|y_1 \dots y_{i-1}| = \sum_{j=1}^{i-1} |y_j| < n_{i-1}$  and  $|y_i| \in n_{i-1}\mathbb{N}$ , we have  $[y_1 \dots y_i]_{n_{i-1}} = y_1 \dots y_{i-1}$  and hence  $\psi^0(\phi^0(y))_i = y_i$ . Conversely,  $\phi^0(\psi^0(x))_i = \psi^0(x)_1 \dots \psi^0(x)_i = x_1 \psi^0(x)_1 \dots \psi^0(x)_i = x_2 \psi^0(x_3) = \dots = x_i$ . Therefore  $\psi^0$  is an inverse for  $\phi^0$ .

The basic open sets in  $Y$  are given by  $Z_Y(w_1, \dots, w_k) = \{y \in Y : y_i = w_i \text{ for } 1 \leq i \leq k\}$ , where  $w_i \in X_i$  and  $s(w_i) = r(w_{i+1})$ .

We calculate

$$\begin{aligned} \phi^0(Z_Y(y_1, \dots, y_k)) &= \{x \in F^0 : x_i = y_1 \dots y_i \text{ for all } i \leq k\} \\ &= \{x \in F^0 : x_k = y_1 \dots y_k\} = Z(y_1 \dots y_k, k). \end{aligned}$$

So  $\psi^0$  is continuous.

Conversely, for  $\mu \in E^{<n_k}$ , express  $\mu = y_1 \dots y_k$ , where  $y_1 = [\mu]_{n_1}$  and  $[\mu]_{n_i} y_{i+1} = [y]_{n_{i+1}}$ . Then  $\psi^0(Z(\mu, k)) = Z_Y(y_1, \dots, y_k)$ . So  $\phi^0$  is continuous. Therefore  $\phi^0$  is a homeomorphism of  $E(\infty)^0$  onto  $F^0$ . It then follows immediately that  $\phi^1 : E(\infty)^1 \rightarrow F^1$  is also a homeomorphism.

We have  $\phi^0(s_{E(\infty)}(e, y)) = s_F(e, \phi^0(y)) = s_F(\phi^1(e, y))$ . We also have  $\phi^0(r_{E(\infty)}(e, y))_i = [ey_1 \dots y_i]_{n_i}$ , so  $\phi^0(r_{E(\infty)}(e, y)) = r_F(e, \phi^0(y)) = r_F(\phi^1(e, y))$ . Therefore  $\phi$  is an isomorphism of topological graphs.  $\square$

We now analyse connectivity in the topological graph  $F$  when  $E$  is finite and strongly connected.

We need to recall some facts from Perron-Frobenius theory for finite strongly connected graphs. Recall (for example from [21, Section 6] with  $k = 1$ ) that the *period*  $\mathcal{P}_E$  of a strongly connected directed graph  $E$  is given by  $\mathcal{P}_E = \gcd\{|\mu| : \mu \in E^*, r(\mu) = s(\mu)\}$ . The group  $\mathcal{P}_E \mathbb{Z}$  is then equal to the subgroup generated by  $\{|\mu| : \mu \in vE^*v\}$  for any vertex  $v$  of  $E$ , and so is equal to  $\{|\mu| - |\nu| : \mu, \nu \in vE^*v\}$  for any  $v$ .

**Lemma 4.2.** *Let  $E$  be a strongly connected finite graph with no sources, and take  $n \in \mathbb{N}$ . There is a map  $C_n : E^0 \times E^0 \rightarrow \mathbb{Z} / \gcd(\mathcal{P}_E, n)\mathbb{Z}$  such that  $C_n(r(\lambda), s(\lambda)) = |\lambda| + \gcd(\mathcal{P}_E, n)\mathbb{Z}$  for all  $\lambda \in E^*$ . There is also an equivalence relation  $\sim_n$  on  $E^0$  such that  $v \sim_n w$  if and only if  $C_n(v, w) = 0$ .*

*Proof.* Fix  $v, w \in E^0$  and  $\mu, \nu \in vE^*w$ . Since  $E$  is strongly connected, there is a path  $\lambda \in wE^*v$ , and then  $\mu\lambda, \nu\lambda \in vE^*v$ . Hence  $|\mu| - |\nu| = |\mu\lambda| - |\nu\lambda| \in \mathcal{P}_E \mathbb{Z} \subseteq \gcd(\mathcal{P}_E, n)\mathbb{Z}$ . So there is a well-defined function  $C_n : \{(v, w) \in E^0 \times E^0 : vE^*w \neq \emptyset\} \rightarrow \mathbb{Z} / \gcd(\mathcal{P}_E, n)\mathbb{Z}$

such that  $C_n(r(\lambda), s(\lambda)) = |\lambda| + \gcd(\mathcal{P}_E, n)\mathbb{Z}$  for all  $\lambda$ . Since  $E$  is strongly connected, the domain of  $C_n$  is all of  $E^0 \times E^0$  as claimed.

Define a relation  $\sim_n$  on  $E^0$  by  $v \sim_n w$  if  $C_n(v, w) = 0$ . We show that  $\sim_n$  is an equivalence relation. We clearly have  $C_n(v, v) = 0$  for all  $v$ , so  $\sim_n$  is reflexive. To see that it is symmetric, suppose that  $C_n(v, w) = 0$ . Then there exists  $\lambda \in vE^*w$  with  $|\lambda| \in \gcd(\mathcal{P}_E, n)\mathbb{Z}$ . Since  $E$  is strongly connected, there exists  $\mu \in wE^*v$ , and then  $\lambda\mu \in vE^*v$ . Hence  $|\lambda\mu| \in \mathcal{P}_E\mathbb{Z}$ . Now  $|\mu| = |\lambda\mu| - |\lambda| \in \mathcal{P}_E\mathbb{Z} \subseteq \gcd(\mathcal{P}_E, n)\mathbb{Z}$ , and so  $C_n(w, v) = 0$  as well. For transitivity, suppose that  $C_n(u, v) = 0$  and  $C_n(v, w) = 0$ . Then there exist  $\mu \in uE^*v$  and  $\nu \in vE^*w$  with  $|\mu|, |\nu| \in \gcd(\mathcal{P}_E, n)\mathbb{Z}$ . So  $\mu\nu \in uE^*w$  satisfies  $|\mu\nu| = |\mu| + |\nu| \in \gcd(\mathcal{P}_E, n)\mathbb{Z}$ , and hence  $C_n(u, w) = 0$  too.  $\square$

**Proposition 4.3.** *Let  $E$  be a strongly connected finite directed graph with no sources. For  $n \in \mathbb{N}$ , the connected components of  $E(n)$  are the sets  $E(n)_\Lambda^0 := \{\mu \in E^{<n} : s(\mu) \in \Lambda\}$  indexed by  $\Lambda \in E^0/\sim_n$ . These connected components are all strongly-connected: if  $\mu, \nu \in E(n)_\Lambda^0$ , then  $\mu E(n)^*\nu \neq \emptyset$ . In particular,  $E(n)$  is strongly connected if and only if  $\gcd(\mathcal{P}_E, n) = 1$ .*

Recall that for  $\lambda = \lambda_1 \dots \lambda_l \in E^*$  and  $\mu \in E^{<n}$  with  $s(\lambda) = r(\mu)$ , we write  $(\lambda, \mu)$  for the corresponding path  $(\lambda_1, [\lambda_2 \dots \lambda_l \mu]_n)(\lambda_2, [\lambda_3 \dots \lambda_l \mu]_n) \dots (\lambda_l, \mu) \in [\lambda\mu]_n E(n)^l \mu$ . In particular, if  $\lambda \in E^l$ , then  $(\lambda, s(\lambda)) \in E(n)^l$ .

We write  $\approx_E$  for the smallest equivalence relation on  $E^0$  such that  $r(e) \approx_E s(e)$  for all  $e \in E^1$ . We call the equivalence classes of  $\approx_E$  the connected components of  $E$ .

*Proof of Proposition 4.3.* Since  $\mu E(n)^*\nu \neq \emptyset$  implies  $\mu \approx_{E(n)} \nu$ , it suffices to show that if  $s(\mu) \sim_n s(\nu)$  then  $\mu E(n)^*\nu \neq \emptyset$ , and that if  $\mu \approx_{E(n)} \nu$ , then  $s(\mu) \sim_n s(\nu)$ .

First suppose that  $s(\mu) \sim_n s(\nu)$ . Since  $E$  has no sources and is strongly connected, it has no sinks, so we can choose  $\alpha = \alpha_1 \dots \alpha_k \in E^*r(\nu)$  such that  $|\alpha\nu| \in n\mathbb{N}$ . It follows that  $C(r(\alpha), s(\nu)) = 0$  and so  $s(\mu) \sim_n r(\alpha)$ . Let  $v := s(\mu)$  and  $w := r(\alpha)$ . Since  $v \sim_n w$ , we have  $|\lambda| + \gcd(\mathcal{P}_E, n)\mathbb{Z} = C_n(v, w) = 0$ , so  $|\lambda| \in \gcd(\mathcal{P}_E, n)\mathbb{Z}$ . Choose  $k$  such that  $k\mathcal{P}_E \equiv \gcd(\mathcal{P}_E, n) \pmod{n}$ . Since  $E$  is strongly connected, we have  $\mathcal{P}_E\mathbb{Z} = \{|\eta| - |\zeta| : \eta, \zeta \in wE^*w\}$ . So there are cycles  $\eta, \zeta \in wE^*w$  such that  $|\eta| - |\zeta| = \mathcal{P}_E$ . In particular,  $|\eta\zeta^{n-1}| = |\eta| - |\zeta| + |\zeta^n| = \mathcal{P}_E + n|\zeta| \equiv \mathcal{P}_E \pmod{n}$ . Hence  $\beta := (\eta\zeta^{n-1})^k \in wE^*w$  satisfies  $|\beta| \equiv k\mathcal{P}_E \pmod{n} \equiv \gcd(\mathcal{P}_E, n) \pmod{n}$ . Choose  $q \in \mathbb{N}$  such that  $qn \geq |\lambda|$ . Since  $|\lambda|$  is divisible by  $n$ , the number  $l := \frac{qn - |\lambda|}{\gcd(\mathcal{P}_E, n)}$  is an integer. Now  $|\lambda\beta^l| \in vE^{jn}w$  for some  $j$ . So (2.1) gives a path  $\tilde{\lambda} \in vE(n)^*w$ . Now  $(\mu, \nu)\tilde{\lambda}(\alpha, \nu) \in \mu E(n)\nu$  as required.

Now suppose that  $\mu \approx_{E(n)} \nu$ . Since  $(\mu, s(\mu)) \in \mu E(n)^*s(\mu)$  and likewise for  $\nu$ , and since  $\approx_{E(n)}^0$  is an equivalence relation, we have  $s(\mu) \approx_{E(n)} s(\nu)$ . So it suffices to show that  $v \approx_{E(n)} w$  implies  $v \sim_n w$  for  $v, w \in E^0$ . By definition of  $\approx_{E(n)}$  it then suffices, by induction, to show that if  $vE(n)^*w \neq \emptyset$ , say  $(\lambda, w) \in vE(n)^*w$ , then  $v \sim_n w$ . By (2.1) we have  $\lambda \in vE^{jn}w$  for some  $j$ . In particular,  $C(v, w) = |\lambda| + \gcd(\mathcal{P}_E, n)\mathbb{Z} = 0 + \gcd(\mathcal{P}_E, n)\mathbb{Z}$  and so  $v \sim_n w$ .  $\square$

Given a sequence  $\omega = (n_k)_{k=1}^\infty$  of natural numbers with  $n_k \mid n_{k+1}$  for all  $k$ , and given  $p \in \mathbb{N}$ , the sequence  $\gcd(p, n_k)$  is nondecreasing and bounded above by  $p$ , so it is eventually constant. We write  $\gcd(p, \omega)$  for its eventual value.

**Lemma 4.4.** *Let  $E$  be a strongly connected finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Fix  $k$  with  $\gcd(\mathcal{P}_E, n_k) = \gcd(\mathcal{P}_E, \omega)$ . For each equivalence class  $\Lambda \in E^0/\sim_{n_k}$ , let*

$X_\Lambda = \bigcup_{\mu \in E^{<n_k}, s(\mu) \in \Lambda} Z(\mu, k)$ . The  $X_\Lambda$  are mutually disjoint and cover  $F^0 = \varprojlim E^{<n_k}$ . Each  $X_\Lambda$  is invariant in the sense of [18, Definition 2.1], and the  $X_\Lambda$  are the minimal nonempty closed invariant subsets of  $F^0$ .

*Proof.* Take  $\Lambda, \Lambda' \in E^0 / \sim_{n_k}$  with  $\Lambda \neq \Lambda'$ . Since  $x \in X_\Lambda$  if and only if  $s(x_k) \in \Lambda$ , it is clear that  $X_\Lambda$  and  $X_{\Lambda'}$  are mutually orthogonal.

To see that each  $X_\Lambda$  is invariant, let  $\mu \in E^{<n_k}$  and  $e \in E^1 r(\mu)$ . Then

$$r_{n_k}(e, \mu) = [e\mu]_{n_k} = \begin{cases} e\mu & \text{if } |e\mu| < n_k \\ r(e) & \text{if } |e\mu| = n_k, \end{cases}$$

so  $s(r_{n_k}(e, \mu)) \in \Lambda$  if and only if  $s(\mu) \in \Lambda$ . Now, take  $(e, x) \in F^1$ . We have  $s(x_k) \in \Lambda$  if and only if  $s(r_{n_k}(e, x_k)) = s([ex_k]_{n_k}) \in \Lambda$ . So  $x \in X_\Lambda$  if and only if  $r_F(e, x) = [ex_k]_{n_k} \in X_\Lambda$ .

For the final assertion, fix  $x = (x_k)_{k=1}^\infty$  and  $y = (y_k)_{k=1}^\infty$  in a given  $X_\Lambda$ . It suffices to show that for every  $k \in \mathbb{N}$ , there exists  $\mu_k \in F^*$  such that  $s_F(\mu_k) = x$  and  $r_F(\mu_k) \in Z(y_k, k)$ . Fix  $k$  such that  $\gcd(\mathcal{P}_E, n_k) = \gcd(\mathcal{P}_E, \omega)$ . Proposition 4.3 implies that the component  $E(n_k)_\Lambda^0$  is strongly connected. So there exists  $\lambda \in E(n_k)^*$  such that  $s_{n_k}(\lambda) = x_k$  and  $r_{n_k}(\lambda) = y_k$ . Say  $\lambda = (\lambda_1, [\lambda_2 \dots \lambda_i x_k]_{n_k}) \dots (\lambda_{i-1}, [\lambda_i x_k]_{n_k})(\lambda_i, x_k)$ . Define  $\mu_i := (\lambda_i, x) \in F^1$  and inductively let  $\mu_j = (\lambda_j, r_F(\mu_{j+1})) \in F^1$  for  $1 \leq j \leq i-1$ . Then  $\mu = \mu_1 \dots \mu_i \in F^i$  and  $s_F(\mu) = s_F(\mu_i) = x$ . By construction,  $(\mu_j)_k = (\lambda_j, [\lambda_{j+1} x_k]_{n_k})$  for each  $1 \leq j \leq i-1$ , so  $r_F(\mu)_k = r_F(\mu_1)_k = r_{n_k}(\lambda_1, [\lambda_2 \dots \lambda_i x_k]_{n_k}) = r_{n_k}(\lambda) = y_k$ , so  $r_F(\mu) \in Z(y_k, k)$ .  $\square$

## 5. UNIQUENESS THEOREMS AND SIMPLICITY

In this section we prove uniqueness theorems for  $\mathcal{T}(E, \omega)$  and  $C^*(E, \omega)$ . Interestingly, in contrast to the uniqueness theorems for directed graph algebras, no gauge-invariance hypothesis or aperiodicity hypotheses are needed in the uniqueness theorem for  $C^*(E, \omega)$  provided that  $n_k \rightarrow \infty$ . To obtain our uniqueness theorem for  $C^*(E, \omega)$  we appeal to Katsura's theory of topological graphs and their  $C^*$ -algebras using the construction of the preceding section.

Our first uniqueness theorem is for  $\mathcal{T}(E, \omega)$ , and follows relatively easily from Fowler and Raeburn's uniqueness theorem [9, Theorem 4.1] for Toeplitz algebras of Hilbert bi-modules.

**Proposition 5.1.** *Let  $E$  be a row-finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Let  $(T, Q, \psi)$  be an  $\omega$ -representation of  $E$  in a  $C^*$ -algebra  $A$ . Then the induced homomorphism  $\pi_{T, Q, \psi} : \mathcal{T}(E, \omega) \rightarrow A$  is injective if and only if*

$$(5.1) \quad \left( Q_{r(\mu)} - \sum_{e \in r(\mu)E^1} T_e T_e^* \right) \psi_{(\mu, k)} \neq 0$$

for all  $k \in \mathbb{N}$  and  $\mu \in E^{<n_k}$ .

*Proof.* Fix  $k \in \mathbb{N}$  and let  $(t_{n_k, (e, \mu)}, q_{n_k, \mu})$  be the universal Toeplitz–Cuntz–Krieger  $E(n_k)$ -family in  $\mathcal{TC}^*(E(n_k)) = \mathcal{T}(E, n_k)$ . Fix  $\mu \in E^{<n_k}$ . The composition  $\varphi_{T, Q, \psi} \circ i_{n_k, \infty}$  carries  $q_{n_k, \mu} - \sum_{(e, \nu) \in \mu E(n_k)^1} t_{n_k, (e, \nu)} t_{n_k, (e, \nu)}^*$  to  $\psi_{(\mu, k)} - \sum_{(e, \nu) \in \mu E(n_k)^1} T_e \psi_{(\nu, k)} T_e^*$ . Applying the relation (3.1) and collecting terms we obtain

$$\varphi_{T, Q, \psi} \circ i_{n_k, \infty} \left( q_{n_k, \mu} - \sum_{(e, \nu) \in \mu E(n_k)^1} t_{n_k, (e, \nu)} t_{n_k, (e, \nu)}^* \right) = \left( Q_{r(\mu)} - \sum_{e \in r(\mu)E^1} T_e T_e^* \right) \psi_{(\mu, k)}.$$

Theorem 4.1 of [9] shows that  $\varphi_{T,Q,\psi} \circ i_{n_k,\infty} : \mathcal{TC}^*(E(n_k)) \rightarrow A$  is injective if and only if  $\varphi_{T,Q,\psi} \circ i_{n_k,\infty} \left( q_{n_k,\mu} - \sum_{(e,\nu) \in \mu E(n_k)^1} t_{n_k,(e,\nu)} t_{n_k,(e,\nu)}^* \right) \neq 0$  for all  $\mu \in E^{<n_k}$ . Since  $i_{n_k,\infty}$  is injective for each  $k$ , the result follows.  $\square$

We now state our main uniqueness result, which characterises the injective homomorphisms of  $C^*(E, \omega)$ .

**Theorem 5.2.** *Let  $E$  be a row-finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Suppose that  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Suppose that  $(S, P, \psi)$  is an  $\omega$ -representation of  $E$ . Then  $\varphi_{S,P,\psi}$  is injective if and only if  $\psi_{(\mu,k)} \neq 0$  for all  $k \in \mathbb{N}$  and  $\mu \in E^{<n_k}$ .*

To prove this theorem, we use Katsura’s results about topological graph  $C^*$ -algebras, and the isomorphism  $C^*(E, \omega) \cong C^*(E(\infty))$  established by Kribs and Solel. The following result follows from the isomorphism  $F \cong E(\infty)$ , and Katsura’s arguments in [17], but a precise description of the isomorphism that we need to use is not provided there, so we give a detailed statement.

**Proposition 5.3.** *Let  $E$  be a row-finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Suppose that  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$ . There is an isomorphism  $\pi : \varinjlim C^*(E(n_k)) \rightarrow C^*(F)$  such that*

$$\pi(j_{n_k,\infty}(p_{n_k,\lambda})) = t_F^0(\chi_{Z(\lambda,k)}), \quad \text{and} \quad \pi(j_{n_k,\infty}(n_k, s_{(e,\lambda)})) = t_F^1(\chi_{\{e\} \times Z(\lambda,k)}),$$

where  $(t_F^0, t_F^1)$  is the universal Cuntz–Krieger pair for  $C^*(F)$ .

*Proof.* For a topological graph  $E$  we denote by  $(t_E^0, t_E^1)$  the universal Cuntz–Krieger pair for  $C^*(E)$ . The argument of Katsura [17, Proposition 2.9] shows that each regular factor map  $m : E \rightarrow F$  of topological graphs  $E$  and  $F$  induces a homomorphism  $\mu_m : C^*(F) \rightarrow C^*(E)$  such that  $\mu_m \circ t_F^i = t_E^i \circ m_*^i$ , for  $i = 0, 1$ .

Let  $j_{n_k,\infty}$  be the universal map from  $C^*(E(n_k))$  into  $\varinjlim C^*(E(n_k))$ . Let  $\psi : F \rightarrow E(\infty)$  be the inverse of the isomorphism of Lemma 5.3. Kribs and Solel define regular factor maps  $m_{k,k+1} : E(n_{k+1}) \rightarrow E(n_k)$  such that  $E(\infty) = \varprojlim E(n_k)$ . For each  $k$ , write  $m_k : E(\infty) \rightarrow E(n_k)$  for the induced factor map. In [19, Theorem 6.3] Kribs and Solel invoke [17, Proposition 4.13] to show that there is an isomorphism  $\rho : \varinjlim (C^*(E(n_k)), j_{k,k+1}) \rightarrow C^*(E(\infty))$ ; it follows from the arguments of [17, Proposition 4.13] that  $\rho \circ j_{k,\infty} = \mu_{n_k}$ . Define  $\pi := \mu_\psi \circ \rho$ . Since  $\psi$  is an isomorphism, so is  $\mu_\psi$ , and

$$\begin{aligned} \pi(j_{n_k,\infty}(p_{n_k,\lambda})) &= \mu_\psi(\rho(j_{n_k,\infty}(p_{n_k,\lambda}))) \\ &= \mu_\psi \circ \mu_{m_k}(p_{n_k,\lambda}) = \mu_\psi(t_{E(\infty)}^0(\chi_{\psi(Z(\lambda,k))})) = t_F^0(\chi_{Z(\lambda,k)}). \end{aligned}$$

A similar calculation gives  $\pi(j_{n_k,\infty}(n_k, s_{(e,\lambda)})) = t_F^1(\chi_{\{e\} \times Z(\lambda,k)})$ .  $\square$

*Proof of Theorem 5.2.* The identifications  $C^*(E, n_k) = C^*(E(n_k))$  induce an isomorphism  $\alpha : C^*(E, \omega) \cong \varinjlim (C^*(E(n_k)), j_{k,k+1})$  such that  $\alpha(\rho_{(\mu,k)}) = j_{k,\infty}(p_\mu)$ . So, if

$$\pi : \varinjlim (C^*(E(n_k)), j_{k,k+1}) \rightarrow C^*(F)$$

is the isomorphism from the proof of Proposition 5.3, we have  $\pi \circ \alpha(\rho_{(\mu,k)}) = t_F^0(\chi_{Z(\mu,k)})$  for all  $k \in \mathbb{N}$ ,  $\mu \in E^{<n_k}$ . Hence  $\phi_{s,p,\psi} \circ \alpha^{-1} \circ \pi^{-1}$  is a homomorphism of  $C^*(F)$  that carries  $t_F^0(\chi_{Z(\mu,k)})$  to  $\psi_{(\mu,k)}$ . Kribs and Solel show that  $E(\infty)$  has no cycles, so Lemma 4.1 shows that  $F$  has no cycles. So [16, Theorem 5.12] implies that  $\psi_{s,p,\psi} \circ \alpha^{-1} \circ \pi^{-1}$  is injective if and only if each  $\psi_{(\mu,k)} \neq 0$ . Since  $\alpha$  and  $\pi$  are isomorphisms, the result follows.  $\square$

We now turn our attention to simplicity of  $C^*(E, \omega)$ . In [19, Section 9], Kribs and Solel provide a necessary and sufficient condition on the topological graph  $E(\infty)$  for  $\varinjlim C^*(E, \omega)$  to be simple. In this section, we consider finite strongly connected graphs, and we employ Perron–Frobenius theory, as well as Katsura’s characterisation of simplicity for  $C^*$ -algebras of topological graphs [18, Theorem 8.12], to improve upon Kribs and Solel’s result to obtain a necessary and sufficient condition in terms of  $E$  and  $\omega$  for simplicity of  $C^*(E, \omega)$  provided that the terms  $n_k$  in  $\omega$  diverge to infinity. (If the  $n_k$  are bounded then they are eventually constant, and then  $C^*(E, \omega) \cong C^*(E(N))$ ; and so simplicity of  $C^*(E, \omega)$  is characterised by [3, Proposition 5.1].)

The following technical result will be useful again later in our analysis of the structure of the factor KMS states on  $\mathcal{TC}^*(E, \omega)$ .

**Lemma 5.4.** *Let  $E$  be a strongly connected finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Fix  $k$  such that  $\gcd(\mathcal{P}_E, n_k) = \gcd(\mathcal{P}_E, \omega)$ . For each equivalence class  $\Lambda \in E^0 / \sim_{n_k}$ , let*

$$Q_{k,\Lambda} := \sum_{\mu \in E^{<n_k}, s(\mu) \in \Lambda} \pi_{(\mu,k)} \in \mathcal{T}(E, \omega).$$

*Then the  $Q_{k,\Lambda}$  are nonzero mutually orthogonal projections, and*

$$\mathcal{T}(E, \omega) = \bigoplus_{\Lambda \in E^0 / \sim_{n_k}} Q_{k,\Lambda} \mathcal{T}(E, \omega) Q_{k,\Lambda}.$$

*The images  $P_{k,\Lambda}$  of the  $Q_{k,\Lambda}$  in the quotient  $C^*(E, \omega)$  are also nonzero, and the direct summands  $P_{k,\Lambda} C^*(E, \omega) P_{k,\Lambda}$  are simple.*

*Proof.* For  $\Lambda \in E^0 / \sim_{n_k}$ , we put

$$\Theta_{k,\Lambda} := \sum_{\mu \in E^{<n_k}, s(\mu) \in \Lambda} \theta_{n_k, \mu} \in \mathcal{T}(E, n_k).$$

The  $\Theta_{k,\Lambda}$  are mutually orthogonal by Proposition 4.3, and nonzero because the generators of  $\mathcal{T}(E, n_k) \cong \mathcal{TC}^*(E(n_k))$  are all nonzero.

We claim that for  $\alpha \in E^{<n_k}$  and  $\mu, \nu \in E^* r(\alpha)$ , we have  $\sum_{\Lambda} Q_{\Lambda} t_{\mu} \theta_{(\alpha,k)} t_{\nu}^* Q_{\Lambda} = t_{\mu} \theta_{(\alpha,k)} t_{\nu}^*$ . Let  $\Lambda$  be the equivalence class of  $\alpha$  under  $\approx_{E(n_k)}$ . Since  $C_{n_k}(s(r_{n_k}(\mu, \alpha)), s(\alpha)) = C_{n_k}(s([\mu\alpha]_{n_k}), s(\alpha)) = 0$ , we have  $s(r_{n_k}(\mu, \alpha)) \in \Lambda$ . Similarly,  $s(r_{n_k}(\nu, \alpha)) \in \Lambda$ . Let  $(t, q)$  be the universal Toeplitz–Cuntz–Krieger  $E(n_k)$ -family in  $\mathcal{TC}^*(E(n_k))$ . We have

$$\begin{aligned} \Theta_{k,\Lambda} t_{n_k, \mu} \theta_{n_k, \alpha} t_{n_k, \nu}^* \Theta_{k,\Lambda} &= \sum_{\eta, \zeta \in E^{<n_k}, s(\eta), s(\zeta) \in \Lambda} (q_{\eta} t_{(\mu, \alpha)} t_{(\nu, \alpha)}^* q_{\zeta}) \\ &= t_{(\mu, \alpha)} t_{(\nu, \alpha)}^* = t_{n_k, \mu} \theta_{n_k, \alpha} t_{n_k, \nu}^*, \end{aligned}$$

and since the  $\Theta_{k,\Lambda}$  are mutually orthogonal, the claim follows.

We now show that each  $i_{n_k, n_{k+1}}(\Theta_{k,\Lambda}) = \Theta_{k+1, \Lambda}$ . We calculate:

$$\begin{aligned} i_{n_k, n_{k+1}}(\Theta_{k,\Lambda}) &= i_{n_k, n_{k+1}} \left( \sum_{\eta \in E^{<n_k}, s(\eta) \in \Lambda} \theta_{n_k, \eta} \right) \\ &= \sum_{\zeta \in E^{<n_{k+1}}, s([\zeta]_{n_k}) \in \Lambda} \theta_{n_{k+1}, \zeta} = \sum_{\zeta \in E^{<n_{k+1}}, s(\zeta) \in \Lambda} \theta_{n_{k+1}, \zeta} = \Theta_{k+1, \Lambda}. \end{aligned}$$



The preceding two paragraphs show that every element of the spanning family for  $\mathcal{T}(E, \omega)$  described in the final statement of Lemma 3.8 belongs to  $Q_{k,\Lambda} \mathcal{T}(E, \omega) Q_{k,\Lambda}$  for some  $\Lambda$ , giving the desired direct-sum decomposition.

To see that the images  $P_{k,\Lambda}$  of the  $Q_{k,\Lambda}$  in  $C^*(E, \omega)$  are nonzero, observe that for each  $\Lambda$ , and any  $v \in \Lambda$ , we have  $P_{k,\Lambda} \geq \rho_{(v,k)} = p_{n_k,v}$ , which is nonzero since all the generators of  $C^*(E(n_k))$  are nonzero. For the assertion about simplicity, observe that the isomorphism  $C^*(E, \omega) \cong C^*(F)$  determined by Proposition 5.3 carries each  $P_{k,\Lambda}$  to  $t_F^1(\chi_{X_\Lambda})$ , where the  $X_\Lambda$  are the minimal invariant subsets of  $F^0$  described in Lemma 4.4. For each  $\Lambda$ , let  $F_\Lambda$  be the topological subgraph of  $F$  given by  $F_\Lambda^0 = X_\Lambda$  and  $F_\Lambda^1 = r_f^{-1}(X_\Lambda)$ . Since  $F_\Lambda^0$  and  $F_\Lambda^1$  are clopen in  $F^0$  and  $F^1$ , there are canonical inclusions  $C(F_\Lambda^0) \hookrightarrow C(F^0)$  and  $C(F_\Lambda^1) \hookrightarrow C(F^1)$ , and it is easy to verify that the universal property of  $C^*(F_\Lambda)$  applied to these inclusions gives surjective homomorphisms  $\iota_\Lambda : C^*(F_\Lambda) \rightarrow t_F^1(\chi_{X_\Lambda}) C^*(F) t_F^1(\chi_{X_\Lambda})$ . Lemma 4.4 shows that each  $F_\Lambda$  is invariant. By [19, Lemma 9.1]  $E(\infty)$  has no loops, so Lemma 4.1 shows that  $F$  also has no loops, and hence each  $F_\Lambda$  has no loops. Hence [18, Theorem 8.12] shows that each  $C^*(F_\Lambda)$  is simple. Hence each  $P_{k,\Lambda} C^*(E, \omega) P_{k,\Lambda} \cong C^*(F_\Lambda)$  is simple.  $\square$

**Corollary 5.5.** *Let  $E$  be a strongly connected finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Suppose that  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Then  $C^*(E, \omega)$  is simple if and only if  $\gcd(\mathcal{P}_E, \omega) = 1$ .*

*Proof.* This follows immediately from the final statement of Lemma 5.4.  $\square$

## 6. KMS STATES

In this section we study the KMS states for the gauge action on  $\mathcal{T}(E, \omega)$ . Throughout this section, if  $X$  is a compact topological space, then  $\mathcal{M}_1^+(X)$  denotes the Choquet simplex of Borel probability measures on  $X$ . We write  $A_E$  for the adjacency matrix  $A_E(v, w) = |vE^1w|$  of a finite graph  $E$ , and  $\rho(A_E)$  for its spectral radius.

The following summarises our main results about KMS states on  $\mathcal{T}(E, \omega)$  and  $C^*(E, \omega)$ .

**Theorem 6.1.** *Let  $E$  be a finite strongly connected graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Let  $\alpha : \mathbb{R} \rightarrow \text{Aut } \mathcal{T}(E, \omega)$  be given by  $\alpha_t = \gamma_{e^{it}}$ .*

- (1) *For  $\beta > \ln \rho(A_E)$  there is an affine isomorphism (described in Corollary 6.15) of  $\mathcal{M}_1^+(\varprojlim E^{<n_k})$  onto the  $\text{KMS}_\beta$ -simplex of  $\mathcal{T}(E, \omega)$ .*
- (2) *There are exactly  $\gcd(\mathcal{P}_E, \omega)$  extremal  $\text{KMS}_{\ln \rho(A_E)}$ -states of  $\mathcal{T}(E, \omega)$  (described explicitly in Theorem 6.16).*
- (3) *For  $\beta < \ln \rho(A_E)$ , there are no  $\text{KMS}_\beta$  states for  $\mathcal{T}(E, \omega)$ .*
- (4) *A  $\text{KMS}_\beta$  state of  $\mathcal{T}(E, \omega)$  factors through  $C^*(E, \omega)$  if and only if  $\beta = \ln \rho(A_E)$ .*

**6.1. A transformation on finite signed Borel measures.** Let  $E$  be a finite directed graph with no sources, and  $\omega = (n_k)_{k=1}^\infty$  a sequence of positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . We consider the Banach space  $\mathcal{M}(\varprojlim E^{<n_k})$  of finite signed measures on the spectrum  $\varprojlim E^{<n_k}$  of the commutative subalgebra of  $C^*(E, \omega)$  described in Section 3. We show that the vertex adjacency matrices  $A_{E(n_k)}$  induce a bounded linear transformation  $A_\omega$  of  $\mathcal{M}(\varprojlim E^{<n_k})$ . We use Perron–Frobenius theory to show that  $\|A_\omega\| = \rho(A_E)$ , and that it always admits a positive eigenmeasure. We provide a condition under which this eigenmeasure is unique up to scalar multiples.

For  $k \geq 1$ , define a map  $p_{n_{k+1}, n_k}^* : \mathcal{M}(E^{< n_{k+1}}) \rightarrow \mathcal{M}(E^{< n_k})$  by  $p_{n_{k+1}, n_k}^*(m)(U) = m(p_{n_{k+1}, n_k}^{-1}(U))$ , where  $U$  is a Borel measurable subset of  $E^{< n_k}$ . Then  $p_{n_{k+1}, n_k}^*$  is linear and the  $(\mathcal{M}(E^{< n_k}), p_{n_{k+1}, n_k}^*)$  form a projective sequence of Banach spaces, giving a Fréchet space  $\varprojlim (\mathcal{M}(E^{< n_k}), p_{n_{k+1}, n_k}^*)$ . We can also form the Banach space  $\mathcal{M}(\varprojlim E^{< n_k})$ . The following lemma describes an injective (but typically not surjective) linear map from the latter into the former; the result must be standard, but it is also easy enough to give a quick proof.

**Lemma 6.2.** *Let  $E$  be a finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . There is a continuous injective linear map  $\iota_\omega : \mathcal{M}(\varprojlim E^{< n_k}) \rightarrow \varprojlim (\mathcal{M}(E^{< n_k}), p_{n_{k+1}, n_k}^*)$  such that  $\iota_\omega(m)_k(\{\tau\}) = m(Z((\tau, k)))$  for all  $m, k, \tau$ .*

*Proof.* For each  $k \geq 1$ , define  $p_{\infty, n_k}^* : \mathcal{M}(\varprojlim E^{< n_k}) \rightarrow \mathcal{M}(E^{< n_k})$  by  $p_{\infty, n_k}^*(m)(\{\tau\}) = m(p_{\infty, n_k}^{-1}(\tau))$ . Then each  $p_{\infty, n_k}^*$  is linear, and we have

$$p_{n_{k+1}, n_k}^*(p_{\infty, n_{k+1}}^*(m))(\{\tau\}) = m(p_{\infty, n_{k+1}}^{-1}(p_{n_{k+1}, n_k}^{-1}(\tau))) = m(p_{\infty, n_k}^{-1}(\tau)) = p_{\infty, n_k}^*(m)(\{\tau\})$$

for all  $k$ . So the universal property of  $\varprojlim (\mathcal{M}(E^{< n_k}), p_{n_{k+1}, n_k}^*)$  implies that there is a continuous map  $\iota_\omega$  such that  $\iota_\omega(m)_k(\tau) = m(Z(\tau, k))$  for all  $m, k, \tau$ . Direct calculation shows that  $\iota_\omega$  is linear.

For injectivity, take  $m \in \mathcal{M}(\varprojlim E^{< n_k})$  with  $\iota_\omega(m) = 0$ . For each  $k \in \mathbb{N}$  and  $\mu \in E^{< n_k}$ , we have  $m(Z(\mu, k)) = \iota_\omega(m)_k(\{\mu\}) = 0$ , and since the  $Z(\mu, k)$  are a basis for  $\varprojlim E^{< n_k}$ , we deduce that  $m = 0$ .  $\square$

*Remark 6.3.* The map  $\iota_\omega$  is typically not surjective. For example, let  $E$  be the directed graph with one vertex  $v$  and one edge  $e$ . Define  $m_0 \in \mathcal{M}(E^0)$  by  $m_0(\{v\}) = 1$ . Let  $n_k = 2^k$  for all  $k$ , and inductively define  $m_k \in \mathcal{M}(E^{< n_k})$  by

$$m_k(\{e^j\}) = 2m_{k-1}(\{e^j\}) \quad \text{and} \quad m_k(\{e^{j+2^{k-1}}\}) = -m_{k-1}(\{e^j\})$$

for  $j \in \{0, \dots, 2^{k-1} - 1\}$ . Then  $(m_k)_{k=1}^\infty \in \varprojlim \mathcal{M}(E^{< n_k})$ , but we have  $m_k(\{v\}) = 2^k \rightarrow \infty$ . For any  $m \in \mathcal{M}(\varprojlim E^{< n_k})$ , we have  $\iota_\omega(m)_k(\{v\}) = m(Z(k, \tau)) \leq m^+(Z(k, \tau))$  for all  $k$ , so the sequence  $\iota_\omega(m)_k(\{v\})$  is bounded. So  $(m_k)_{k=1}^\infty$  does not belong to the range of  $\iota_\omega$ .

In what follows, if  $m \in \mathcal{M}(\varprojlim E^{< n_k})$ , we will frequently write  $m_{n_k}$  for  $\iota_\omega(m)_k \in \mathcal{M}(E^{< n_k})$ . We also regard the adjacency matrix  $A_{E(n_k)}$  as a linear transformation of the finite-dimensional vector space  $\mathcal{M}(E^{< n_k}) \cong \mathbb{R}^{E^{< n_k}}$ . We show how the  $A_{E(n_k)}$  induce a linear transformation of  $\varprojlim \mathcal{M}(E^{< n_k})$ .

**Lemma 6.4.** *Let  $E$  be a finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . For  $k \in \mathbb{N}$  let  $A_{n_k} := A_{E(n_k)}$ , regarded as a linear transformation of  $\mathcal{M}(E^{< n_k})$ . For  $m \in \mathcal{M}(E^{< n_k})$ , we have*

$$(6.1) \quad (A_{n_k} m)(\{\mu\}) = \begin{cases} m(\{\mu_2 \dots \mu_{|\mu|}\}) & \text{if } \mu \in E^{< n_k} \setminus E^0 \\ \sum_{e\nu \in \mu E^{n_k}} m(\{\nu\}) & \text{if } \mu \in E^0, \end{cases}$$

and

$$(6.2) \quad A_{n_{k-1}}(p_{n_k, n_{k-1}}^*(m)) = p_{n_k, n_{k-1}}^*(A_{n_k}(m)).$$

*Proof.* We write  $\{\delta_{\mu,k} : \mu \in E^{<n_k}\}$  for the basis of Dirac measures on  $E^{<n_k}$ . We have

$$\begin{aligned} A_{n_k}(\delta_{\mu,k}) &= \sum_{\nu \in E^{<n_k}} A_{n_k}(\nu, \mu) \delta_{\nu,k} \\ &= \sum_{\nu \in E^{<n_k}} |\nu E(n_k)^1 \mu| \delta_{\nu,k} = \begin{cases} \delta_{\mu_2 \dots \mu_{|\mu|}, k} & \text{if } \mu \in E^{<n_k} \setminus E^0 \\ \sum_{e\nu \in \mu E^{n_k}} \delta_{\nu,k} & \text{if } \mu \in E^0. \end{cases} \end{aligned}$$

Now (6.1) follows from linearity.

To prove (6.2), first consider  $\mu \in E^{<n_{k-1}} \setminus E^0$ . We have

$$\begin{aligned} A_{n_{k-1}}(p_{n_k, n_{k-1}}^*(m))(\{\mu\}) &= p_{n_k, n_{k-1}}^*(m)(\{\mu_2 \dots \mu_{|\mu|}\}) = \sum_{\tau \in E^{<n_k}, [\tau]_{n_{k-1}} = \mu_2 \dots \mu_{|\mu|}} m(\{\tau\}) \\ &= \sum_{\eta \in E^{<n_k}, [\eta]_{n_{k-1}} = \mu} A_{n_k}(m)(\{\eta\}) = p_{n_k, n_{k-1}}^*(A_{n_k}(m))(\{\mu\}). \end{aligned}$$

Now consider  $\mu = v \in E^0$ . We have

$$\begin{aligned} A_{n_{k-1}}(p_{n_k, n_{k-1}}^*(m))(\{v\}) &= \sum_{e\tau \in v E^{n_{k-1}}} p_{n_k, n_{k-1}}^*(m)(\{\tau\}) = \sum_{\substack{e \in v E^1, \lambda \in s(e) E^{<n_k} \\ |e\lambda| \in n_{k-1} \mathbb{N}}} m(\{\lambda\}) \\ &= \sum_{\lambda \in E^{<n_k}, [\lambda]_{n_{k-1}} = v} A_{n_k}(m)(\{\lambda\}) = p_{n_k, n_{k-1}}^*(A_{n_k}(m))(\{v\}). \quad \square \end{aligned}$$

**Proposition 6.5.** *Let  $E$  be a finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . For  $k \in \mathbb{N}$  let  $A_{n_k} := A_{E(n_k)}$ , regarded as a linear transformation of  $\mathcal{M}(E^{<n_k})$ . There is a linear transformation  $A_\omega$  of  $\varprojlim \mathcal{M}(E^{<n_k})$  given by  $A_\omega m = (A_{n_1} m_1, A_{n_2} m_2, \dots)$ . The inclusion  $\iota_\omega$  of Lemma 6.2 satisfies*

$$A_\omega(\iota_\omega(\mathcal{M}(\varprojlim E^{<n_k}))) \subseteq \iota_\omega(\mathcal{M}(\varprojlim E^{<n_k}))$$

*Proof.* Fix  $(m_1, m_2, \dots) \in \varprojlim \mathcal{M}(E^{<n_k})$ . By Lemma 6.4 we have  $p_{n_k, n_{k-1}}^*(A_{n_k}(m_{n_k})) = A_{n_{k-1}}(p_{n_k, n_{k-1}}^*(m_{n_k})) = A_{n_{k-1}} m_{n_{k-1}}$ , so  $(A_{n_1} m_1, A_{n_2} m_2, \dots) \in \varprojlim \mathcal{M}(E^{<n_k})$ . The universal property of  $\varprojlim \mathcal{M}(E^{<n_k})$  gives a continuous map  $A_\omega : \varprojlim \mathcal{M}(E^{<n_k}) \rightarrow \varprojlim \mathcal{M}(E^{<n_k})$  satisfying  $A_\omega m = (A_{n_1} m_1, A_{n_2} m_2, \dots)$ . It is clear that  $A_\omega$  is linear.

By Lemma 6.4, we have  $p_{n_k, n_{k-1}}^*(A_{n_k} m_{n_k}^+) = A_{n_{k-1}} m_{n_{k-1}}^+$ . So by [4, Theorem 2.2], there is a positive Borel measure  $M^+$  on  $\varprojlim E^{<n_k}$  such that  $M^+(Z(\mu, k)) = (A_{n_k} m_{n_k}^+)(\{\mu\})$  for all  $k \in \mathbb{N}$  and  $\mu \in E^{<n_k}$ . Similarly, there is a positive Borel measure  $M^-$  on  $\varprojlim E^{<n_k}$  such that  $M^-(Z(\mu, k)) = (A_{n_k} m_{n_k}^-)(\{\mu\})$  for  $\mu \in E^{<n_k}$ . Now  $A_\omega \iota_\omega(m) = \iota_\omega(M^+ - M^-)$  belongs to the range of  $\iota_\omega$ .  $\square$

For calculations later, we will want to understand the transformation  $A_\omega$  in terms of the measures of cylinder sets.

**Lemma 6.6.** *Let  $E$  be a finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . For  $m \in \mathcal{M}(\varprojlim E^{<n_k})$ ,*

$k \in \mathbb{N}$  and  $\mu \in E^{<n_k}$ , the transformation  $A_\omega$  of Proposition 6.5 satisfies

$$\begin{aligned} (A_\omega m)(Z(\mu, k)) &= \begin{cases} m(Z(\mu_2 \dots \mu_{|\mu|}, k)) & \text{if } \mu \in E^{<n_k} \setminus E^0 \\ \sum_{e\nu \in \mu E^{n_k}} m(Z(\nu, k)) & \text{if } \mu \in E^0 \end{cases} \\ &= \sum_{\nu \in E^{<n_k}} |\mu E(n_k)^1 \nu| m(Z(\nu, k)). \end{aligned}$$

*Proof.* Since  $A_\omega m(Z(\mu, k)) = A_\omega m(p_{\infty, n_k}^{-1}(\{\mu\})) = A_{n_k} m_{n_k}(\{\mu\})$ , the result follows from Lemma 6.4.  $\square$

We now show that  $A_\omega$  admits a positive eigenmeasure and also that the norm of  $A_\omega$ , as an operator on the Banach space  $\mathcal{M}(\varprojlim E^{<n_k})$ , is  $\rho(A_E)$ . Recall that the unimodular Perron-Frobenius eigenvector of an irreducible nonnegative matrix  $A$  is its unique positive eigenvector with unit 1-norm.

**Proposition 6.7.** *Let  $E$  be a finite strongly connected directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Let  $x^E$  be the unimodular Perron-Frobenius eigenvector of  $A_E$ . The transformation  $A_\omega$  of Proposition 6.5 admits a positive eigenmeasure  $m$  such that*

$$(6.3) \quad m(Z(\mu, k)) = \frac{1}{n_k} \rho(A_E)^{-|\mu|} x_{s(\mu)}^E \quad \text{for all } \mu \in E^{<n_k},$$

and the corresponding eigenvalue is  $\rho(A_E)$ , and is equal to the operator norm of  $A_\omega$  as a transformation of  $\mathcal{M}(\varprojlim E^{<n_k})$ .

*Proof.* To see that (6.3) specifies an element  $m \in \mathcal{M}(\varprojlim E^{<n_k})$ , define measures  $m_k$  by  $m_k(\{\mu\}) := \frac{1}{n_k} \rho(A_E)^{-|\mu|} x_{s(\mu)}^E$  for  $\mu \in E^{<n_k}$ . Let  $a_k := n_{k+1}/n_k$  for each  $k$ . Using at the fifth equality that  $A_E^j x^E = \rho(A_E)^j x^E$  for all  $j$ , we calculate

$$\begin{aligned} p_{n_{k+1}, n_k}^*(m_{n_{k+1}})(\{\mu\}) &= \sum_{\tau \in E^{<n_{k+1}}, [\tau]_{n_k} = \mu} \frac{1}{n_{k+1}} \rho(A_E)^{-|\tau|} x_{s(\tau)}^E \\ &= \sum_{j=0}^{a_k-1} \frac{1}{n_k} \rho(A_E)^{|\mu|} \sum_{\lambda \in s(\mu) E^{jn_k}} \frac{1}{a_k} \rho(A_E)^{-jn_k} x_{s(\lambda)}^E \\ &= \sum_{j=0}^{a_k-1} \frac{1}{n_k} \rho(A_E)^{|\mu|} \frac{1}{a_k} \rho(A_E)^{-jn_k} \sum_{w \in E^0} A_E^{jn_k}(s(\mu), w) x_w^E \\ &= \sum_{j=0}^{a_k-1} \frac{1}{n_k} \rho(A_E)^{|\mu|} \frac{1}{a_k} \rho(A_E)^{-jn_k} (A_E^{jn_k} x^E)_{s(\mu)} \\ &= \sum_{j=0}^{a_k-1} \frac{1}{n_k} \rho(A_E)^{|\mu|} \frac{1}{a_k} x_{s(\mu)}^E = \frac{1}{n_k} \rho(A_E)^{-|\mu|} x_{s(\mu)}^E = m_{n_k}(\{\mu\}). \end{aligned}$$

Now [4, Theorem 2.2] implies that there is a positive measure  $m$  on  $\varprojlim E^{<n_k}$  satisfying (6.3).

To see that  $m$  is an eigenmeasure for  $A_\omega$  with eigenvalue  $\rho(A_E)$ , observe that for  $\mu \in E^{<n_k} \setminus E^0$ , we have

$$(A_\omega m)(Z(\mu, k)) = m(Z(\mu_2 \dots \mu_{|\mu|}, k)) = \frac{1}{n_k} \rho(A_E)^{-|\mu|+1} x_{s(\mu)}^E = \rho(A_E) m(Z(\mu, k)),$$

and for  $v \in E^0$ , we have

$$\begin{aligned} (A_\omega m)(Z(v, k)) &= \sum_{e \in vE^1, \tau \in s(e)E^{n_k-1}} \frac{1}{n_k} \rho(A_E)^{-|\tau|} x_{s(\tau)}^E = \frac{1}{n_k} \sum_{w \in E^0} \sum_{\lambda \in vE^{n_k}w} \rho(A_E)^{-|\lambda|+1} x_w^E \\ &= \frac{1}{n_k} (A^{n_k} \rho(A_E)^{-n_k+1} x_w^E)_v = \frac{1}{n_k} \rho(A_E) x_v^E. \end{aligned}$$

So  $m$  is an eigenmeasure for  $A_\omega$  with corresponding eigenvalue  $\rho(A_E)$ . It follows immediately that  $\|A_\omega\| \geq \rho(A_E)$ . For the reverse inequality, take  $m \in \mathcal{M}(\varprojlim E^{<n_k})$  and consider its Jordan decomposition  $m = m^+ - m^-$ . Since  $A_\omega$  is linear, we have  $A_\omega m^+ - A_\omega m^- = A_\omega m$ , and since the  $A_{n_k}$  are positive matrices, the measures  $A_\omega m^\pm$  are positive measures. So the Jordan Decomposition Theorem implies that  $A_\omega m^+ \geq (A_\omega m)^+$  and  $A_\omega m^- \geq (A_\omega m)^-$ . So

$$\begin{aligned} \|A_\omega\| &= \sup_{\|m\|=1} \|A_\omega m\| = \sup_{\|m\|=1} ((A_\omega m)^+(\varprojlim E^{<n_k}) + (A_\omega m)^-(\varprojlim E^{<n_k})) \\ &\leq \sup_{\|m\|=1} ((A_\omega m^+)(\varprojlim E^{<n_k}) + (A_\omega m^-)(\varprojlim E^{<n_k})) \\ &= \sup_{\|m\|=1} ((A_1 m_1^+)(E^0) + (A_1 m_1^-)(E^0)) \leq \sup_{\|m\|=1} (\rho(A_E) m_1^+(E^0) + \rho(A_E) m_1^-(E^0)) \\ &= \rho(A_E) \sup_{\|m\|=1} (m^+(\varprojlim E^{<n_k}) + m^-(\varprojlim E^{<n_k})) = \rho(A_E). \end{aligned} \quad \square$$

We now show that if  $E$  is strongly connected and  $\gcd(\mathcal{P}_E, \omega) = 1$ , then the measure  $m$  of the preceding proposition is the only positive probability measure that is an eigenmeasure for the transformation  $A_\omega$ .

**Lemma 6.8.** *Let  $E$  be a finite strongly connected directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Let  $m$  be the measure of Proposition 6.7, and fix  $k$  such that  $\gcd(\mathcal{P}_E, n_k) = \gcd(\mathcal{P}_E, \omega)$ .*

- (1) *Let  $\sim_{n_k}$  be the equivalence relation on  $E^0$  of Lemma 4.2. For  $\Lambda \in E^0/\sim_{n_k}$ , let  $X_\Lambda = \bigcup_{\mu \in E^{<n_k}, s(\mu) \in \Lambda} Z(\mu, k) \subseteq \varprojlim E^{<n_k}$ , and define  $m^\Lambda \in \mathcal{M}(\varprojlim E^{<n_k})$  by  $m^\Lambda(U) := \frac{1}{m(X_\Lambda)} m(U \cap X_\Lambda)$ . Then each  $m^\Lambda$  is a normalised eigenmeasure for  $A_\omega$  with eigenvalue  $\rho(A_E)$ .*
- (2) *For each  $l \geq k$ , and for each  $\Lambda \in E^0/\sim_{n_k}$ , the block  $A_{n_l}^\Lambda \in M_{E^{<n_l}\Lambda}(\mathbb{Z})$  of  $A_{n_l}$  is an irreducible matrix. We have  $\rho(A_{n_l}^\Lambda) = \rho(A_E)$  and  $m_{n_l}^\Lambda = (m^\Lambda(Z(\mu, l)))_{\mu \in E^{<n_l}}$  is the unimodular Perron–Frobenius eigenvector of  $A_{n_l}^\Lambda$ .*
- (3) *Every positive eigenmeasure for  $A_\omega$  is a convex combination of the  $m^\Lambda$ .*

*Proof.* (1) Proposition 6.7 shows that  $m$  is an eigenmeasure with  $A_\omega m = \rho(A_E) m$ . Lemma 4.4 shows that each  $\mathcal{M}(X_\Lambda) \subseteq \mathcal{M}(\varprojlim E^{<n_k})$  is invariant for  $A_\omega$ , and it follows that  $A_\omega m^\Lambda = \rho(A_E) m^\Lambda$  for each  $\Lambda$ .

(2) For each  $l \geq k$  and each  $\Lambda \in E^0/\sim_{n_k}$ , the matrix  $A_{n_l}^\Lambda$  is irreducible by Proposition 4.3. By definition of  $A_\omega$ , we have  $A_\omega \chi_{Z(\mu, l)} = \sum_\nu A_{n_l}^\Lambda(\nu, \mu) \chi_{Z(\nu, l)}$ , and so (1) shows

that  $A_{n_l}^\Lambda m_{n_l}^\Lambda = \rho(A_E) m_{n_l}^\Lambda$ . The Perron-Frobenius theorem [28, Theorem 1.5] implies that every entry of the Perron-Frobenius eigenvector of the irreducible matrix  $A_E$  is nonzero, and so (6.3) shows that  $m_{n_l}^\Lambda$  is the unimodular Perron-Frobenius eigenvector of  $A_{n_l}^\Lambda$ , and so its eigenvalue  $\rho(A_E)$  is equal to  $\rho(A_{n_l}^\Lambda)$ .

(3) Suppose that  $m' \in \mathcal{M}^+(\varprojlim E^{<n_k})$  and  $z \in \mathbb{C}$  satisfy  $A_\omega m' = z m'$ . Then in particular  $A_{n_l}(m')_{n_l}^\Lambda = (z m')_{n_l}^\Lambda$  for each  $l \geq k$  and  $\Lambda \in E^0 / \sim_{n_l}$ . Since each  $A_{n_l}^\Lambda$  is irreducible, this forces  $z = \rho(A_{n_l}) = \rho(A_E)$ , and  $(m')_{n_k}^\Lambda$  is a scalar multiple of  $m_{n_l}^\Lambda$ , so  $m' = \sum_\Lambda t_\Lambda m_{n_l}^\Lambda$ . Since the supports of the  $m_{n_l}^\Lambda$  are disjoint and  $m'$  is positive, the  $t_\Lambda$  are positive, and their sum is 1 because  $m'$  and the  $m_{n_l}^\Lambda$  are normalised. Since this is true for all  $l$ , continuity implies that  $m' = \sum_\Lambda t_\Lambda m^\Lambda$ .  $\square$

**Lemma 6.9.** *Let  $E$  be a finite strongly connected directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Suppose that  $s > 0$  and  $m \in \mathcal{M}^+(\varprojlim E^{<n_k})$  satisfy  $A_\omega m \leq s m$ . Then  $s \geq \rho(A_E)$ . Moreover,  $s = \rho(A_E)$  if and only if  $A_\omega m = s m$ .*

*Proof.* Since  $A_\omega m \leq s m$ , we have  $A_E m_1 \leq s m_1$ , and since  $A_E$  is irreducible, the subinvariance theorem [28, Theorem 1.6] implies that  $s \geq \rho(A_E)$ .

Suppose that  $s = \rho(A_E)$ . For  $k$  such that  $\gcd(\mathcal{P}_E, n_k) = \gcd(\mathcal{P}_E, \omega)$ , the matrix  $A_{n_k}^\Lambda$  is irreducible by Proposition 4.3, so the forward implication of the last assertion of [28, Theorem 1.6] implies that  $A_{n_k}^\Lambda m_{n_k} = \rho(A_{n_k}^\Lambda) m_{n_k}$ . Since  $\rho(A_{n_k}^\Lambda) = \rho(A_E)$  for all  $k$  by part (2) of Lemma 6.8, we deduce that  $A_{n_k} m_{n_k} = \rho(A_E) m_{n_k}$  for all  $k$ . So  $A_\omega m = \rho(A_E) m$ .

Now suppose that  $A_\omega m = s m$ . Then part (3) of Lemma 6.8 gives  $s = \rho(A_E)$ .  $\square$

**6.2. Characterising KMS states.** We characterise the  $\text{KMS}_\beta$ -states for the gauge action on  $\mathcal{T}(E, \omega)$  in terms of their values at spanning elements  $t_\mu \pi_{(\alpha, k)} t_\nu^*$ . We describe a subinvariance condition on the measure  $m^\phi$  on  $\varprojlim E^{<n_k}$  induced by a KMS state  $\phi$ . We also show that a KMS state factors through  $C^*(E, \omega)$  if and only if this subinvariance condition is invariance. Our approach follows the general program of [22], but is by now quite streamlined.

**Theorem 6.10.** *Let  $E$  be a finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Let  $\alpha : \mathbb{R} \rightarrow \text{Aut } \mathcal{T}(E, \omega)$  be given by  $\alpha_t = \gamma_{e^{it}}$ . Let  $\beta \in \mathbb{R}$ .*

(1) *A state  $\phi$  of  $\mathcal{T}(E, \omega)$  is a  $\text{KMS}_\beta$  state for  $\alpha$  if and only if*

$$(6.4) \quad \phi(t_\mu \pi_{(\tau, k)} t_\nu^*) = \delta_{\mu, \nu} e^{-\beta |\mu|} \phi(\pi_{(\tau, k)})$$

*for all  $k \in \mathbb{N}$ , all  $\tau \in E^{<n_k}$  and all  $\mu, \nu \in E^* r(\tau)$ .*

(2) *Suppose that  $\phi$  is a  $\text{KMS}_\beta$  state for  $(\mathcal{T}(E, \omega), \alpha)$ , and let  $m^\phi$  be the measure on  $\varprojlim E^{<n_k}$  such that  $m^\phi(Z(\mu, k)) = \phi(\pi_{(\mu, k)})$  for  $\mu \in E^{<n_k}$ . Then  $m^\phi$  is a probability measure and satisfies the subinvariance relation  $A_\omega m^\phi \leq e^\beta m^\phi$ .*

(3) *A  $\text{KMS}_\beta$  state  $\phi$  of  $(\mathcal{T}(E, \omega), \alpha)$  factors through  $C^*(E, \omega)$  if and only if  $A_\omega m^\phi = e^\beta m^\phi$ .*

*Proof.* (1) Suppose that  $\phi$  is KMS. Then  $\phi$  is  $\alpha$ -invariant—by [2, Proposition 5.33] if  $\beta \neq 0$ , or by definition if  $\beta = 0$ —and so also  $\gamma$ -invariant, and then

$$\phi(t_\mu \pi_{(\tau, k)} t_\nu^*) = \int_{\mathbb{T}} \phi(\gamma_z(t_\mu \pi_{(\tau, k)} t_\nu^*)) dz = \int_{\mathbb{T}} z^{|\mu| - |\nu|} dz \phi(t_\mu \pi_{(\tau, k)} t_\nu^*),$$

which is zero if  $|\mu| \neq |\nu|$ . If  $|\mu| = |\nu|$ , then the KMS condition gives

$$\phi(t_\mu \pi_{(\tau,k)} t_\nu^*) = e^{-\beta|\mu|} \phi(t_\nu^* t_\mu \pi_{(\tau,k)}) = \delta_{\mu,\nu} e^{-\beta|\mu|} \phi(\pi_{(\tau,k)}).$$

Now suppose that  $\phi$  satisfies (6.4). Then the argument of [12, Proposition 2.1(a)] shows that  $\phi$  is KMS.

(2) We have  $m^\phi \geq 0$  because  $\phi$  is a state. To see that  $m^\phi$  is a probability measure, just observe that  $\phi$  restricts to a state of  $\pi(C_0(\varprojlim E^{<n}))$ , and so  $m^\phi$  is a probability measure by the Riesz representation theorem. To see that it satisfies the subinvariance condition, we calculate:

$$\begin{aligned} \sum_{e \in r(\mu)E^1} \phi(t_e t_e^* \pi_{(\mu,k)}) &= \sum_{e \in r(\mu)E^1} e^{-\beta} \phi(t_e^* \pi_{(\mu,k)} t_e) \\ &= e^{-\beta} \begin{cases} \phi(\pi_{(\mu_2 \dots \mu_{|\mu|}, k)} t_{\mu_1}^* t_{\mu_1}) & \text{if } \mu \notin E^0 \\ \sum_{e \nu \in r(\nu)E^{n_k}} \phi(\pi_{(\nu,k)} t_e^* t_e) & \text{if } \mu \in E^0 \end{cases} \\ &= e^{-\beta} \begin{cases} m^\phi(Z(\mu_2 \dots \mu_{|\mu|}, k)) & \text{if } \mu \notin E^0 \\ \sum_{e \nu \in r(\nu)E^{n_k}} m^\phi(Z(\nu, k)) & \text{if } \mu \in E^0 \end{cases} \\ &= e^{-\beta} A_\omega m^\phi(Z(\mu, k)) \end{aligned} \tag{6.5}$$

by Lemma 6.6. Hence each

$$e^\beta m^\phi(Z(\mu, k)) = e^\beta \phi(\pi_{(\mu,k)}) = e^\beta \phi(p_{r(\mu)} \pi_{(\mu,k)}) \geq \sum_{e \in r(\mu)E^1} e^\beta \phi(t_e t_e^* \pi_{(\mu,k)}) = A_\omega m^\phi(Z(\mu, k)).$$

(3) Recall that  $C^*(E, \omega)$  is the quotient of  $\mathcal{T}(E, \omega)$  by the ideal generated by the projections  $q_v - \sum_{e \in vE^1} t_e t_e^*$ ,  $v \in E^0$ . Thus by Lemma 2.2 of [12] it suffices to check that  $\phi(q_v - \sum_{e \in vE^1} t_e t_e^*) = 0$  for all  $v$  if and only if  $A_\omega m^\phi = e^\beta m^\phi$ . For each  $v \in E^0$  and  $k \geq 1$ , we have

$$q_v - \sum_{e \in vE^1} t_e t_e^* = \sum_{\mu \in vE^{<n_k}} \left( q_{r(\mu)} - \sum_{e \in r(\mu)E^1} t_e t_e^* \right) \pi_{(\mu,k)}.$$

Since each term in the last sum is nonnegative,  $\phi(q_v - \sum_{e \in vE^1} t_e t_e^*) = 0$  for each  $v$  if and only if  $\phi((q_{r(\mu)} - \sum_{e \in r(\mu)E^1} t_e t_e^*) \pi_{(\mu,k)}) = 0$  for all  $\mu \in E^{<n_k}$ . By (6.5) we have

$$\begin{aligned} \phi\left(\left(q_{r(\mu)} - \sum_{e \in r(\mu)E^1} t_e t_e^*\right) \pi_{(\mu,k)}\right) &= \phi\left(\pi_{(\mu,k)} - \sum_{e \in r(\mu)E^1} t_e t_e^* \pi_{(\mu,k)}\right) \\ &= e^\beta m^\phi(Z(\mu, k)) - (A_\omega m^\phi)(Z(\mu, k)), \end{aligned}$$

and the result follows.  $\square$

**6.3. Constructing KMS states at large inverse temperatures.** In this section, for each measure  $m$  satisfying the subinvariance relation of Theorem 6.10(2) we construct a KMS state of  $\mathcal{T}(E, \omega)$  that induces  $m$ . We also show that positive subinvariant measures  $m$  are in bijection with positive Borel probability measures on  $\varprojlim E^{<n_k}$ . Let

$$E^* \times_{E^0} \varprojlim E^{<n_k} := \{(\lambda, x) : \lambda \in E^*, x \in \varprojlim E^{<n_k}, s(\lambda) = r(x_1)\}.$$

Let  $\{h_{\lambda,x} : (\lambda, x) \in E^* \times_{E^0} \varprojlim E^{<n_k}\}$  be the canonical basis for  $\ell^2(E^* \times_{E^0} \varprojlim E^{<n_k})$ .

It is not hard to check using a sequential argument that  $x \mapsto (r_{n_i}(\lambda, x_i))_{i=1}^\infty$  is continuous from  $\varprojlim E^{<n_k}$  to  $\varprojlim E^{<n_k}$ . So for a finite graph  $E$  and each  $\lambda \in E^*$ , there is a map  $\alpha_\lambda : C(\varprojlim E^{<n_k}) \rightarrow C(\varprojlim E^{<n_k})$  such that

$$\alpha_\lambda(\chi_{Z(\mu,k)})(x) := \begin{cases} \chi_{Z(\mu,k)}((r_{n_i}(\lambda, x_i))_{i=1}^\infty) & \text{if } s(\lambda) = r(x) \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 6.11.** *Let  $E$  be a row-finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . There is a representation  $\varsigma : \mathcal{T}(E, \omega) \rightarrow \mathcal{B}(\ell^2(E^* \times_{E^0} \varprojlim E^{<n_k}))$  such that for  $e \in E^1$  and  $v \in E^0$ ,*

$$\varsigma(t_e)h_{\lambda,x} = \delta_{r(\lambda),s(e)}h_{e\lambda,x} \quad \text{and} \quad \varsigma(q_v)h_{\lambda,x} = \delta_{r(\lambda),v}h_{\lambda,x},$$

and such that for  $\mu \in E^{<n_k}$ , we have  $\varsigma(\pi_{(\mu,k)})h_{\lambda,x} = \alpha_\lambda(\chi_{Z(\mu,k)})(x)h_{\lambda,x}$

*Proof.* We aim to invoke the universal property of  $\mathcal{T}(E, \omega)$ . It is routine to check that the formulas given for  $\varsigma(t_e)$  and  $\varsigma(q_v)$  define a Toeplitz–Cuntz–Krieger  $E$ -family  $(T, Q)$  in  $\mathcal{B}(\ell^2(E^* \times_{E^0} \varprojlim E^{<n_k}))$ .

Likewise, for each  $k$ , the formula given for the  $\varsigma(\pi_{(\mu,k)})$  determines mutually orthogonal projections indexed by  $\mu \in E^{<n_k}$  and satisfying  $\varsigma(\pi_{(\mu,k)}) = \sum_{\nu \in E^{<n_{k+1}}, [\nu]_{n_k} = \mu} \varsigma(\pi_{(\nu,k+1)})$ , so they determine a homomorphism  $\tilde{\varsigma} : C(\varprojlim E^{<n_k}) \rightarrow \mathcal{B}(\ell^2(E^* \times_{E^0} \varprojlim E^{<n_k}))$ .

We show that  $(T, Q, \tilde{\varsigma})$  is a Toeplitz  $\omega$ -representation of  $E$ . Take  $e \in E^1$  and  $\mu \in E^{<n_k}$  and suppose that  $\mu = e\mu'$ . For any  $(\lambda, x) \in E^* \times_{E^0} \varprojlim E^{<n_k}$ , we have

$$T_e^* \tilde{\varsigma}_{(\mu,k)} h_{\lambda,x} = T_e^* \alpha_\lambda(\chi_{Z(\mu,k)})(x) h_{\lambda,x} = \begin{cases} \alpha_\lambda(\chi_{Z(\mu,k)})(x) h_{\lambda',x} & \text{if } \lambda = e\lambda' \\ 0 & \text{otherwise.} \end{cases}$$

Also,

$$\tilde{\varsigma}_{(\mu',k)} T_e^* h_{\lambda,x} = \begin{cases} \varsigma_{(\mu',k)} h_{\lambda',x} & \text{if } \lambda = e\lambda' \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \alpha_{\lambda'}(\chi_{Z(\mu',k)})(x) h_{\lambda',x} & \text{if } \lambda = e\lambda' \\ 0 & \text{otherwise.} \end{cases}$$

If  $\lambda \neq e\lambda'$  then both  $T_e^* \tilde{\varsigma}_{(\mu,k)} h_{\lambda,x}$  and  $\tilde{\varsigma}_{(\mu',k)} T_e^* h_{\lambda,x}$  are zero, so suppose that  $\lambda = e\lambda'$ . Then  $\alpha_\lambda(\chi_{Z(\mu,k)})(x) = \chi_{Z(\mu,k)}(r_{n_i}(\lambda, x_i)_{i=1}^\infty) = 1$  if and only if  $\alpha_{\lambda'}(\chi_{Z(\mu',k)})(x) = 1$  as well; so  $T_e^* \tilde{\varsigma}_{(\mu,k)} = \tilde{\varsigma}_{(\mu',k)} T_e^*$ .

Now let  $v = r(e)$ , and observe that

$$T_e^* \tilde{\varsigma}_{(v,k)} h_{\lambda,x} = \begin{cases} \alpha_\lambda(\chi_{Z(v,k)})(x) h_{\lambda',x} & \text{if } \lambda = e\lambda' \\ 0 & \text{otherwise,} \end{cases}$$

while

$$\sum_{e\tau \in E^{n_k}} \tilde{\varsigma}_{(\tau,k)} T_e^* h_{\lambda,x} = \begin{cases} \sum_{e\tau \in E^{n_k}} \alpha_{\lambda'}(\chi_{Z(\tau,k)})(x) h_{\lambda',x} & \text{if } \lambda = e\lambda' \\ 0 & \text{otherwise.} \end{cases}$$

Again, if  $\lambda \neq e\lambda'$ , then both expressions are zero, so we suppose that  $\lambda = e\lambda'$ . We have  $\alpha_\lambda(\chi_{Z(v,k)})(x) = 1$  if and only if  $r(\lambda) = v$  and  $|\lambda x_i| \in n_i \mathbb{N}$  for large  $i$ . Also,  $\sum_{e\tau \in E^{n_k}} \alpha_{\lambda'}(\chi_{Z(\tau,k)})(x) = 1$  if and only if  $[\lambda' x_i]_{n_i} \in E^{n_i-1}$  for large  $i$ , which is equivalent to  $|\lambda' x_i| \equiv n_i - 1 \pmod{n_i}$  for large  $i$ , and so  $T_e^* \tilde{\varsigma}_{(v,k)} h_{\lambda,x} = \sum_{e\tau \in E^{n_k}} \tilde{\varsigma}_{(\tau,k)} T_e^* h_{\lambda,x}$  as required.

Finally, suppose that  $\mu \neq e\mu'$  and  $\mu \neq r(e)$ . We immediately see that  $T_e^* \tilde{\varsigma}_{(\mu,k)} = 0$  if  $\mu \in E^0 \setminus r(e)$ . If  $\mu \notin E^0$ , then  $\mu_1 \neq e$ , so that  $\tilde{\varsigma}_{(\mu,k)}$  is the projection onto a subspace



of  $\overline{\text{span}}\{h_{\lambda,x} : (\lambda x_i)_1 = \mu_1 \text{ for large } i\}$ , which is orthogonal to the projection  $T_e T_e^*$  onto  $\overline{\text{span}}\{h_{\lambda,x} : \lambda_1 = e\}$ .

We have now established that  $(T, Q, \tilde{\varsigma})$  is an  $\omega$ -representation, and so the universal property of  $\mathcal{T}(E, \omega)$  gives the desired homomorphism  $\varsigma$ .  $\square$

The following technical result will help in our construction of KMS states.

**Lemma 6.12.** *Let  $E$  be a strongly connected finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Take  $\beta > \ln \rho(A_E)$ . The series  $\sum_{j=0}^\infty e^{-\beta j} A_\omega^j$  converges in norm to an inverse for  $1 - e^{-\beta} A_\omega$ . For  $\varepsilon \in \mathcal{M}^+(\varprojlim E^{<n_k})$  and  $\tau \in E^{<n_k}$ ,*

$$(1 - e^{-\beta} A_\omega)^{-1}(\varepsilon)(Z(\tau, k)) = \sum_{(\lambda, \nu) \in \tau E(n_k)^*} e^{-\beta|\lambda|} \varepsilon(Z(\nu, k)).$$

*Proof.* Proposition 6.7 gives  $\|A_\omega\| = \rho(A_E)$ . Since  $\beta > \ln \rho(A_E)$ , we have  $\|e^{-\beta} A_\omega\| < 1$ , and so  $\sum_{j=0}^\infty e^{-\beta j} A_\omega^j$  converges in operator norm to  $(1 - e^{-\beta} A_\omega)^{-1}$ .

Now take  $\tau \in E^{<n_k}$ . Using Lemma 6.6 at the second equality, we calculate

$$\begin{aligned} (1 - e^{-\beta} A_\omega)^{-1}(\varepsilon)(Z(\tau, k)) &= \sum_{j=0}^\infty e^{-\beta j} (A_\omega^j \varepsilon)(Z(\tau, k)) \\ &= \sum_{j=0}^\infty \sum_{\nu \in E^{<n_k}} e^{-\beta j} |\tau E(n_k)^j \nu| \varepsilon(Z(\nu, k)) \\ &= \sum_{j=0}^\infty \sum_{(\lambda, \nu) \in \tau E(n_k)^j} e^{-\beta j} \varepsilon(Z(\nu, k)) \\ &= \sum_{(\lambda, \nu) \in \tau E(n_k)^*} e^{-\beta|\lambda|} \varepsilon(Z(\nu, k)). \end{aligned} \quad \square$$

We can now construct a KMS state for each measure that satisfies the subinvariance relation in Theorem 6.10(2).

**Proposition 6.13.** *Let  $E$  be a strongly connected finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Take  $\beta > \ln \rho(A_E)$ . Suppose that  $m \in \mathcal{M}_1^+(\varprojlim E^{<n_k})$  satisfies  $A_\omega m \leq e^\beta m$ . Then there is a  $\text{KMS}_\beta$  state  $\phi_m$  of  $(\mathcal{T}(E, \omega), \alpha)$  satisfying*

$$(6.6) \quad \phi_m(t_\mu \pi_{(\tau, k)} t_\nu^*) = \delta_{\mu, \nu} e^{-\beta|\mu|} m(Z(\tau, k))$$

for all  $\tau \in E^{<n_k}$  and all  $\mu, \nu \in E^* r(\tau)$ .

*Proof.* Let  $\varepsilon := (1 - e^{-\beta} A_\omega)m$ . Since  $m$  is subinvariant,  $\varepsilon$  is a positive measure on  $\varprojlim E^{<n_k}$ . Let  $\varsigma : \mathcal{T}(E, \omega) \rightarrow \mathcal{B}(\ell^2(E^* \times_{E^0} \varprojlim E^{<n_k}))$  be the representation of Proposition 6.11. We aim to define  $\phi_m$  by

$$(6.7) \quad \phi_m(a) = \sum_{\lambda \in E^*} e^{-\beta|\lambda|} \int_{x \in \varprojlim E^{<n_k}} \chi_{Z(s(\lambda), 1)}(x) (\varsigma(a) h_{\lambda, x} \mid h_{\lambda, x}) d\varepsilon(x).$$

We first show that for  $a \in \mathcal{T}(E, \omega)$ , the function  $f_a : E^* \times_{E^0} \varprojlim E^{<n_k} \rightarrow \mathbb{C}$  given by  $f_a(\lambda, x) = (\varsigma(a)h_{\lambda,x} \mid h_{\lambda,x})$  is integrable. First consider  $a = t_\mu \pi_{(\tau,k)} t_\mu^*$ . We have

$$(6.8) \quad \begin{aligned} (\varsigma(t_\mu \pi_{(\tau,k)} t_\mu^*) h_{\lambda,x} \mid h_{\lambda,x}) &= (\varsigma(\pi_{(\tau,k)} t_\mu^*) h_{\lambda,x} \mid \varsigma(\pi_{(\tau,k)} t_\mu^*) h_{\lambda,x}) \\ &= \begin{cases} \alpha_{\lambda'}(\chi_{Z(\tau,k)})(x) & \text{if } \lambda = \nu \lambda' = \mu \lambda' \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

So  $f_a$  is the characteristic function of the clopen set  $\bigsqcup \{Z(\tau, k) : \tau \in E^{<n_k}, [\lambda\tau]_{n_k} = \mu\}$ , and hence integrable. Consequently  $f_a$  is integrable for  $a \in \text{span}\{t_\mu \pi_{(\tau,k)} t_\mu^*\}$ . Now as in [15, Lemma 10.1(b)], for  $a \in \mathcal{T}(E, \omega)$  is a pointwise limit of integrable functions and hence itself integrable as claimed.

Since each  $Z(s(\lambda), 1)$  is measurable, the functions  $\chi_{Z(s(\lambda), 1)} f_a$  are also integrable. Since  $f_a(\lambda, x) \leq \|a\|$  for all  $(\lambda, x)$ , we have  $\int_{\varprojlim E^{<n_k}} \chi_{Z(s(\lambda), 1)} f_a(\lambda, x) d\mu(x) < \|a\|$ . Since  $\beta > \ln \rho(A_E)$ , Lemma 6.12 implies that  $\sum_{\lambda \in E^{*v}} e^{-\beta|\lambda|}$  is convergent for each  $v$ , and so the series on the right-hand side of (6.7) is bounded above by the convergent series  $\sum_{v \in E^0} \sum_{\lambda \in E^{*v}} e^{-\beta|\lambda|} \|a\|$ , and hence itself convergent. So there is a bounded linear map  $\phi_m : \mathcal{T}(E, \omega) \rightarrow \mathbb{C}$  satisfying (6.7).

This  $\phi_m$  is positive because  $f_{a^*a}$  is positive-valued. We check that  $\phi_m$  is a state. We use Lemma 6.12 at the penultimate equality to calculate

$$\begin{aligned} \phi_m(1) &= \sum_{\lambda \in E^*} e^{-\beta|\lambda|} \int_{x \in \varprojlim E^{<n_k}} \chi_{Z(s(\lambda), 1)}(x) d\varepsilon(x) \\ &= \sum_{\lambda \in E^*} e^{-\beta|\lambda|} \varepsilon(Z(s(\lambda), 1)) = \sum_{w \in E^0} m(Z(w, 1)) = 1. \end{aligned}$$

Since  $\mu \lambda' = \nu \lambda'$  forces  $\mu = \nu$ , we have  $\phi_m(t_\mu \pi_{(\tau,k)} t_\mu^*) = 0$  if  $\mu \neq \nu$ . Moreover, each

$$(\varsigma(t_\mu \pi_{(\tau,k)} t_\mu^*) h_{\lambda,x} \mid h_{\lambda,x}) = \|\varsigma(\pi_{(\tau,k)} t_\mu^*) h_{\lambda,x}\|^2 = \begin{cases} \alpha_{\lambda'}(\chi_{Z(\tau,k)})(x) & \text{if } \lambda = \mu \lambda' \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \phi_m(t_\mu \pi_{(\tau,k)} t_\mu^*) &= \sum_{\mu \lambda' \in E^*} e^{-\beta|\mu \lambda'|} \int_{x \in \varprojlim E^{<n_k}} \alpha_{\lambda'}(\chi_{Z(\tau,k)})(x) d\varepsilon(x) \\ &= e^{\beta|\mu|} \sum_{\lambda' \in s(\mu)E^*} e^{-\beta|\lambda'|} \int_{x \in Z(s(\lambda'), 1)} \chi_{Z(\tau,k)}((r_{n_i}(\lambda', x_i))_{i=1}^\infty) d\varepsilon(x) \\ &= e^{\beta|\mu|} \sum_{\lambda' \in s(\mu)E^*} e^{-\beta|\lambda'|} \varepsilon(\{x : r_{n_k}(\lambda', x_k) = \tau\}) \\ &= e^{\beta|\mu|} \sum_{(\lambda', \nu) \in \tau E^{(n_k)^*}} e^{-\beta|\lambda'|} \varepsilon(Z(\nu, k)) \\ &= e^{\beta|\mu|} m(Z(\tau, k)), \end{aligned}$$

which is (6.6). Putting  $\mu = r(\tau)$  gives  $\phi_m(\pi_{(\tau,k)}) = m(Z(\tau, k))$ , and so  $\phi_m$  also satisfies (6.4), and is therefore KMS by Theorem 6.10(1).  $\square$

**Theorem 6.14.** *Let  $E$  be a strongly connected finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Let  $\alpha : \mathbb{R} \rightarrow \text{Aut}(\mathcal{T}(E, \omega))$  be given by  $\alpha_t = \gamma_{e^{it}}$ . Take  $\beta > \ln \rho(A_E)$ .*

- (1) *Take  $\varepsilon \in \mathcal{M}^+(\varprojlim E^{<n_k})$ . For each  $x \in \varprojlim E^{<n_k}$ , the series  $\sum_{\mu \in E^{*r}(x)} e^{-\beta|\mu|}$  converges; we write  $y(x)$  for its limit. We have  $(1 - e^{-\beta}A_\omega)^{-1}\varepsilon \in \mathcal{M}_1^+(\varprojlim E^{<n_k})$  if and only if*

$$\int_{x \in \varprojlim E^{<n_k}} y(x) d\varepsilon(x) = 1.$$

- (2) *Suppose that  $\varepsilon \in \mathcal{M}^+(\varprojlim E^{<n_k})$  satisfies  $\int_{\varprojlim E^{<n_k}} y(x) d\varepsilon(x) = 1$ , and define  $m := (1 - e^{-\beta}A_\omega)^{-1}\varepsilon$ . There is a  $\text{KMS}_\beta$  state  $\phi_\varepsilon$  of  $(\mathcal{T}(E, \omega), \alpha)$  such that*

$$(6.9) \quad \phi_\varepsilon(t_\mu \pi_{(\tau, k)} t_\nu^*) = \delta_{\mu, \nu} e^{-\beta|\mu|} m(Z(\tau, k)).$$

- (3) *The map  $\varepsilon \mapsto \phi_\varepsilon$  is an affine isomorphism of*

$$\Omega_\beta := \{\varepsilon \in \mathcal{M}^+(\varprojlim E^{<n_k}) : \int y(x) d\varepsilon(x) = 1\}$$

*onto the simplex of  $\text{KMS}_\beta$  states of  $(\mathcal{T}(E, \omega), \alpha)$ . The inverse of this isomorphism takes a  $\text{KMS}_\beta$  state  $\phi$  to  $(1 - e^{-\beta}A_\omega)m^\phi$ .*

*Proof.* (1) The series  $\sum_{j=0}^\infty (e^{-\beta j} A_\omega^j) \varepsilon$  converges to  $m := (1 - e^{-\beta}A_\omega)^{-1}\varepsilon$  because  $\beta > \ln \rho(A_E)$ . This shows that  $m \geq 0$ .

Using Lemma 6.12, we fix  $k$  and calculate

$$\begin{aligned} m(\varprojlim E^{<n_k}) &= \sum_{(\lambda, \nu) \in E(n_k)^*} e^{-\beta|\lambda|} \varepsilon(Z(\nu, k)) = \sum_{\nu \in E^{<n_k}} \sum_{\lambda \in E^{*r}(\nu)} e^{-\beta|\lambda|} \varepsilon(Z(\nu, k)) \\ &= \sum_{\nu \in E^{<n_k}} \int_{x \in Z(\nu, k)} y(x) d\varepsilon(x) = \int_{x \in \varprojlim E^{<n_k}} y(x) d\varepsilon(x). \end{aligned}$$

- (2) We claim that  $A_\omega m \leq e^\beta m$ . We calculate

$$A_\omega m = A_\omega \left( \sum_{j=0}^\infty e^{-\beta j} A_\omega^j \right) \varepsilon = e^\beta \left( \sum_{j=1}^\infty e^{-\beta j} A_\omega^j \right) \varepsilon \leq e^\beta \left( \sum_{j=0}^\infty e^{-\beta j} A_\omega^j \right) \varepsilon = e^\beta m.$$

Now Proposition 6.13 gives a  $\text{KMS}_\beta$  state  $\phi_\varepsilon$  satisfying (6.9).

(3) We claim that every  $\text{KMS}_\beta$  state  $\phi$  has the form  $\phi_\varepsilon$ . Fix a  $\text{KMS}_\beta$  state  $\phi$ , and let  $m^\phi$  be the measure such that  $m^\phi(Z(\mu, k)) = \phi(\pi_{(\mu, k)})$ . By part (2),  $m^\phi$  is a subinvariant probability measure. Let  $\varepsilon := (1 - e^{-\beta}A_\omega)^{-1}m^\phi$ . Then  $m^\phi = (1 - e^{-\beta}A_\omega)\varepsilon$  by construction, and comparing (6.9) with (6.4) shows that  $\phi = \phi_\varepsilon$ .

The formula (6.9) also shows that the map  $F : \varepsilon \rightarrow \phi_\varepsilon$  is injective and weak\*-continuous from  $\Omega_\beta$  to the state space of  $\mathcal{T}(E, \omega)$ . We have just seen that it is surjective onto the  $\text{KMS}_\beta$  simplex, which is compact since  $C^*(E, \omega)$  is unital. Hence  $F$  is a homeomorphism of  $\Omega_\beta$  onto the  $\text{KMS}_\beta$  simplex. The formula (6.7) shows that  $F$  is affine, and the formula for the inverse follows from our proof of surjectivity in the preceding paragraph.  $\square$

**Corollary 6.15.** *Let  $E$  be a strongly connected finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Let  $\alpha : \mathbb{R} \rightarrow \text{Aut}(\mathcal{T}(E, \omega))$  be given by  $\alpha_t = \gamma_{e^{it}}$ . Take  $\beta > \ln \rho(A_E)$ . Let  $y$  be as in part (1) of Theorem 6.14. The map  $m \mapsto \phi_{y^{-1}m}$  is an affine isomorphism of  $\mathcal{M}_1^+(\varprojlim E^{<n_k})$  onto the  $\text{KMS}_\beta$ -simplex of  $(\mathcal{T}(E, \omega), \alpha)$ .*

*Proof.* Since  $y$  takes strictly positive values and is bounded, the map  $m \mapsto y^{-1}m$  is an affine isomorphism of  $\mathcal{M}_1^+(\varprojlim E^{<n_k})$  onto  $\Omega_\beta$ , so the result follows from Theorem 6.14(3).  $\square$

**6.4. KMS states at the critical temperature.** We show that the extreme KMS states at the critical temperature  $\ln \rho(A_E)$  are indexed by the equivalence classes  $E^0/\sim_{n_k}$  of Lemma 4.2 for any  $k$  such that  $\gcd(\mathcal{P}_E, n_k) = \gcd(\mathcal{P}_E, \omega)$ .

**Theorem 6.16.** *Let  $E$  be a strongly connected finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$ . Fix  $k$  such that  $\gcd(\mathcal{P}_E, n_k) = \gcd(\mathcal{P}_E, \omega)$ , and let  $\sim_{n_k}$  be the equivalence relation on  $E^0$  of Lemma 4.2. Let  $\alpha : \mathbb{R} \rightarrow \text{Aut}(\mathcal{T}(E, \omega))$  be given by  $\alpha_t = \gamma_{e^{it}}$ . Let  $x^E$  be the unimodular Perron–Frobenius eigenvector of  $A_E$ .*

(1) *For each  $\Lambda \in E^0/\sim_{n_k}$ , there is a  $\text{KMS}_{\ln \rho(A_E)}$  state  $\phi^\Lambda$  for  $(\mathcal{T}(E, \omega), \alpha)$  satisfying*

$$(6.10) \quad \phi^\Lambda(t_\mu \pi_{(\tau, k)} t_\nu^*) = \chi_\Lambda(s(\tau)) \delta_{\mu, \nu} \frac{1}{\sum_{v \in \Lambda} x_v^E} \rho(A_E)^{-|\mu| - |\tau|} x_{s(\tau)}^E.$$

*This is the unique  $\text{KMS}_{\ln \rho(A_E)}$  state for  $(\mathcal{T}(E, \omega), \alpha)$  satisfying  $\phi^\Lambda(\pi_{(v, k)}) = 0$  for all  $v \in E^0 \setminus \Lambda$ , and it factors through a  $\text{KMS}_{\ln \rho(A_E)}$  state  $\bar{\phi}^\Lambda$  of  $(C^*(E, \omega), \alpha)$ .*

(2) *The states  $\bar{\phi}^\Lambda$  are the extreme points of the  $\text{KMS}_{\ln \rho(A_E)}$ -simplex of  $(C^*(E, \omega), \alpha)$ , and there are no  $\text{KMS}_\beta$ -states for  $(C^*(E, \omega), \alpha)$  for any  $\beta \neq \ln \rho(A_E)$ .*

*Proof.* (1) Fix  $\Lambda \in E^0/\sim_{n_k}$ . We first prove the existence of a  $\text{KMS}_{\ln \rho(A_E)}$  state satisfying (6.10). For each  $l \geq k$ , let  $E(n_l)_\Lambda$  be the component of  $E(n_l)$  with vertices  $E^{<n_l} \Lambda$ . Theorem 4.3(a) of [14] shows that there is a unique KMS state  $\phi_l^\Lambda$  of  $C^*(E(n_l)) = C^*(E, n_l)$  that vanishes on  $\varepsilon_\mu$  for  $\mu \in E^{<n_l}(E^0 \setminus \Lambda)$ . Since each  $\phi_{l+1}^\Lambda$  must restrict to a KMS state of  $C^*(E, n_l)$ , the  $\phi_l^\Lambda$  are compatible under the inclusions  $C^*(E, n_l) \hookrightarrow C^*(E, n_{l+1})$ . So continuity yields a state  $\phi^\Lambda$  on  $C^*(E, \omega)$  that agrees with each  $\phi_{n_l}^\Lambda$  on the image of  $C^*(E, n_l)$ , and hence satisfies (6.10). It follows that  $\phi^\Lambda(\pi_{(v, k)}) = 0$  for all  $v \in E^0 \setminus \Lambda$ . Uniqueness follows from uniqueness of the  $\phi_l^\Lambda$ . Theorem 6.10(3) shows that  $\phi^\Lambda$  factors through  $(C^*(E, \omega), \alpha)$ .

(2) The  $\phi^\Lambda$  are linearly independent, and so are the extreme points of the convex set they generate. So it suffices to show that every KMS state of  $C^*(E, \omega)$  is a convex combination of the  $\phi^\Lambda$ . Suppose that  $\psi$  is a  $\text{KMS}_\beta$  state of  $(C^*(E, \omega), \alpha)$ . Let  $q : \mathcal{T}(E, \omega) \rightarrow C^*(E, \omega)$  be the quotient map. Theorem 6.10(3) implies that  $A_\omega m^{\psi \circ q} = e^\beta m^{\psi \circ q}$ . Hence Lemma 6.8(3) shows that  $m^{\psi \circ q}$  is a convex combination  $m^{\psi \circ q} = \sum_\Lambda t_\Lambda m^\Lambda$  of the  $m^\Lambda$ . It then follows from Theorem 6.10(3) that  $\psi \circ q = \sum_\Lambda t_\Lambda \bar{\phi}^\Lambda$ . Theorem 6.10(3) combined with Lemma 6.8(3) shows that there are no KMS states for  $C^*(E, \omega)$  at any other inverse temperature.  $\square$

*Proof of Theorem 6.1.* Item (1) follows from Corollary 6.15 and item (2) follows from Theorem 6.16.

For item (4), recall that Theorem 6.10(3) implies that a  $\text{KMS}_\beta$  state  $\phi$  factors through  $C^*(E, \omega)$  if and only if  $A_\omega m^\phi = e^{-\beta} m^\phi$ . If  $\phi$  factors through  $C^*(E, \omega)$ , then  $m^\phi$  is a positive eigenmeasure for  $A_\omega$  and Lemma 6.8 gives  $\beta = \ln \rho(A_E)$ . On the other hand, if  $\beta = \ln \rho(A_E)$ , then Theorem 6.10(2) gives  $A_\omega m^\phi \leq \rho(A_E) m^\phi$ , and then Lemma 6.9 forces equality.

Finally, for (3), suppose that  $\phi$  is a  $\text{KMS}_\beta$  state of  $(\mathcal{T}(E, \omega), \alpha)$ . Then Theorem 6.10(2) implies that  $A_\omega m^\phi \leq e^\beta m^\phi$ , and then Lemma 6.9 gives  $e^\beta \geq \rho(A_E)$  and hence  $\beta \geq \ln \rho(A_E)$ .  $\square$

We deduce that simplicity of  $C^*(E, \omega)$  is reflected by the existence of a unique KMS state for the gauge action.

**Proposition 6.17.** *Let  $E$  be a strongly connected finite directed graph with no sources, and take a sequence  $\omega = (n_k)_{k=1}^\infty$  of nonzero positive integers such that  $n_k \mid n_{k+1}$  for all  $k$  and  $n_k \rightarrow \infty$ . Let  $\alpha : \mathbb{R} \rightarrow \text{Aut}(\mathcal{T}(E, \omega))$  be given by  $\alpha_t = \gamma_{e^{it}}$ . The following are equivalent*

- (1)  $\gcd(\mathcal{P}_E, \omega) = 1$ ;
- (2)  $C^*(E, \omega)$  is simple;
- (3) there is a unique KMS state for  $(C^*(E, \omega), \alpha)$  and the state (6.10) factors through this state; and
- (4) the state (6.10) is a factor state.

*Proof.* Corollary 5.5 gives (1)  $\iff$  (2), and Theorem 6.16 gives (1)  $\implies$  (3). To establish (3)  $\implies$  (4), suppose that  $\phi$  factors through the unique KMS state of  $(C^*(E, \omega), \alpha)$ . Then it is an extreme point of the KMS simplex and hence a factor state by [2, Theorem 5.3.30(3)].

For (4)  $\implies$  (1) let  $\phi$  be the state given by (6.10) and suppose that  $\phi$  is a factor state for  $\mathcal{T}C^*(E, \omega)$ . Fix  $k$  such that  $\gcd(\mathcal{P}_E, n_k) = \gcd(\mathcal{P}, \omega)$ . Recall the equivalence relation  $\sim_{n_k}$  of Lemma 4.2 and the projections  $Q_{k, \Lambda}$  of Lemma 5.4. We have  $\phi(\pi_{(\mu, k)}) = \frac{1}{n_k} \rho(A_E)^{-|\mu|} x_{s(\mu)}^E \neq 0$  for all  $\mu$  because the Perron–Frobenius eigenvector has strictly positive entries. So each  $\phi(Q_{k, \Lambda}) \neq 0$ . So the GNS representation  $\pi_\phi$  is also nonzero on the  $Q_{k, \Lambda}$ . Lemma 5.4 implies that the  $Q_{k, \Lambda}$  are central in  $\mathcal{T}(E, \omega)$ , and so the  $\pi_\phi(Q_{k, \Lambda})$  are mutually orthogonal central projections in  $\pi_\phi(\mathcal{T}(E, \omega))''$ . Since  $\phi$  is a factor state, it follows that there is only one equivalence class  $\Lambda$  for  $\sim_{n_k}$ , and so  $\gcd(\mathcal{P}_E, \omega) = 1$ .  $\square$

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